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$U_h(\mathfrak{g})$ invariant quantization of coadjoint orbits and vector bundles over them ☆

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Abstract

Let M be a coadjoint semisimple orbit of a simple Lie group G. Let $U_h(\mathfrak{g})$ be a quantum group corresponding to G. We construct a universal family of $U_h(\mathfrak{g})$ invariant quantizations of the sheaf of functions on M and describe all such quantizations. We also describe all two parameter $U_h(\mathfrak{g})$ invariant quantizations on M, which can be considered as $U_h(\mathfrak{g})$ invariant quantizations of the Kirillov–Kostant–Souriau (KKS) Poisson bracket on M. We also consider how those quantizations relate to the natural polarizations of M with respect to the KKS bracket. Using polarizations, we quantize the sheaves of sections of vector bundles on M as one- and two-sided $U_h(\mathfrak{g})$ invariant modules over a quantized function sheaf. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let G be a simple Lie group with Lie algebra $\mathfrak{g}$, M a semisimple coadjoint orbit of G, i.e., the orbit of G passing through a semisimple element in the coadjoint representation $\mathfrak{g}^*$. Let $\mathcal{A}$ be the sheaf of functions on M. It may be the sheaf of smooth, analytic, or algebraic functions. The universal enveloping algebra $U(\mathfrak{g})$ acts on the sections of $\mathcal{A}$ and the multiplication in $\mathcal{A}$ is $U(\mathfrak{g})$ invariant. Let $U_h(\mathfrak{g})$ denote a quantum group that is a deformation of the $U(\mathfrak{g})$ as a bialgebra. In the paper we [4], considered the following problems.

1. Does there exists a $U_h(\mathfrak{g})$ invariant deformation quantization of $\mathcal{A}$, i.e., a quantization, $\mathcal{A}_h$, having a $U_h(\mathfrak{g})$ invariant multiplication?

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2. Does there exists a two parameter (double) $U_h(\mathfrak{g})$ invariant quantization, $\mathcal{A}_{t,h}$, such that $\mathcal{A}_{t,0}$ is a $U(\mathfrak{g})$ invariant quantization of $\mathcal{A}$ with Poisson bracket being the Kirillov–Kostant–Souriau (KKS) Poisson bracket on M? Note that $\mathcal{A}_{t,h}$ can be considered as a $U_h(\mathfrak{g})$ invariant quantization of the KKS Poisson bracket on M.

In [4], we have classified the Poisson brackets admissible for one and two parameter quantizations. An admissible Poisson bracket for one parameter quantization is the same as a Poisson bracket making M into a Poisson manifold with Poisson action of G, where G is considered to be the Poisson–Lie group with Poisson structure defined by the r-matrix related to $U_h(\mathfrak{g})$. An admissible Poisson bracket for two parameter quantization, in addition, must be compatible with the KKS bracket on M.

We have shown that all semisimple orbits have admissible Poisson brackets for one parameter quantization and almost all such brackets can be quantized. In [4], we called an orbit having a Poisson bracket admissible for a two parameter quantization *a good orbit*. We have classified the good semisimple orbits for all simple $\mathfrak{g}$ and shown that in case $\mathfrak{g} \neq \mathrm{sl}(n)$ all the good orbits can be quantized. In the case $\mathfrak{g} = \mathrm{sl}(n)$ all semisimple orbits are good but in [4] we did not prove the existence of their double quantization.

In this paper, we give a complete description of one and two parameter $U_h(\mathfrak{g})$ invariant quantizations on M. We show that for each semisimple orbit M there exists a universal family of quantizations of $\mathcal{A}$. This family is given by a family of multiplications

$$m_{f,h} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}[[h]], \quad f \in X,$$

where X is the manifold of all admissible Poisson brackets on M. The universality means that any one parameter $U_h(\mathfrak{g})$ invariant quantization of $\mathcal{A}$ is given by the multiplication of the form $m_{f(h),h}$, where $f(h)$ is a formal path in X, and two different paths give non-equivalent quantizations. As a consequence, we obtain that any admissible Poisson bracket on M can be quantized.

There is the analogous description for two parameter quantizations on any good orbit. In particular, we prove that any good orbit (including all semisimple orbits in $\mathrm{sl}(n)^*$) admits a two parameter $U_h(\mathfrak{g})$ invariant quantization.

Further, we consider the natural polarizations on M with respect to the KKS Poisson bracket. We show that all $U_h(\mathfrak{g})$ invariant quantizations on M being restricted to functions constant along a polarization have a standard form. In some cases, when M is a coadjoint orbit of a real Lie group G, the polarizations define complex structures on M. In such cases the sheaf of functions constant along polarization specializes to the sheaf of holomorphic (or anti-holomorphic) functions on M. So, we obtain that in the real case any quantization of smooth functions on M induced a unique quantization of the sub-sheaves of holomorphic and anti-holomorphic functions on M.

In the paper, we also consider the quantization of G invariant vector bundles. We identify such a bundle with the sheaf of its smooth sections. Let $\mathcal{A}_h$ be a $U_h(\mathfrak{g})$ invariant quantization of the sheaf of smooth functions on M. Under the quantization of a vector bundle V on M with respect to $\mathcal{A}_h$ we mean the sheaf $V[[h]]$ endowed with a structure of $U_h(\mathfrak{g})$ invariant left (right, two-sided) $\mathcal{A}_h$ module. Using a complex polarization, we show that for any

$U_h(\mathfrak{g})$ invariant quantization $\mathcal{A}_h$ and any G invariant vector bundle V there exists a $U_h(\mathfrak{g})$ invariant quantization of V as a left $\mathcal{A}_h$ module. Moreover, we show that there exists a special quantization, $\mathcal{A}_h$, such that any vector bundle V admits a quantization as a two-sided module with respect to $\mathcal{A}_h$. Note that the papers [3,14], where the algebra of global holomorphic sections of linear vector bundles on flag varieties was quantized, relate to the problem of quantizing vector bundles.

The paper is organized as follows. In Section 2, we recall some facts on quantum groups essential for our approach to $U_h(\mathfrak{g})$ invariant quantization. In particular, we define a quantum group, $U_h(\mathfrak{g})$, for any classical r-matrix r and show that the problem of constructing $U_h(\mathfrak{g})$ invariant quantization is equivalent to the problem of constructing $U(\mathfrak{g})$ invariant Φ_h associative quantization, where $\Phi_h \in U(\mathfrak{g})^{\otimes 3}[[h]]$ defines the Drinfeld associativity constraint (see [8,9]). Thus, a $U(\mathfrak{g})$ invariant Φ_h associative quantization defines a $U_h(\mathfrak{g})$ invariant quantization for all quantum groups associated with different r. In this section, we also define φ-brackets, which are infinitesimal parts of $U(\mathfrak{g})$ invariant Φ_h associative quantizations. We show that a Poisson bracket admissible for $U_h(\mathfrak{g})$ invariant quantization is the difference of a φ-bracket and the bracket induced on M by the r-matrix associated with $U_h(\mathfrak{g})$.

In Section 3, we give a classification of φ- and good brackets on semisimple orbits. In particular, we give a description of the variety of those brackets, which is more detailed than in [4]. We are needed in this description in the following sections. In Section 4, we consider Poisson cohomologies of some parameterized complexes, which we use in Sections 5 and 6 for proving the existence of the universal one and two parameter quantizations. In Section 7, we consider the natural polarizations on M and prove that any $U(\mathfrak{g})$ invariant Φ_h associative quantization is trivial when restricted to the sheaf of functions constant along polarization. Note that up to Section 7, we assume that G is a complex Lie group, M a complex subvariety in g^*, and $\mathcal{A}$ the sheaf of complex analytic functions on M. In Section 8, we specify our results to a real Lie group G. We consider complex structures corresponding to polarizations and use them to construct the quantizations of G invariant vector bundles.

2. Preliminaries

2.1. Quantum groups

We will consider quantum groups in sense of Drinfeld [8] as deformed universal enveloping algebras. If $U(\mathfrak{g})$ is the universal enveloping algebra of a complex Lie algebra $\mathfrak{g}$, then the quantum group (or quantized universal enveloping algebra) corresponding to $U(\mathfrak{g})$ is a topological Hopf algebra, $U_h(\mathfrak{g})$, over $\mathbb{C}[[h]]$, isomorphic to $U(\mathfrak{g})[[h]]$ as a topological $\mathbb{C}[[h]]$ module and such that $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) = U(\mathfrak{g})$ as a Hopf algebra over $\mathbb{C}$. In particular, the deformed co-multiplication in $U_h(\mathfrak{g})$ has the form

$$\Delta_h = \Delta + h\Delta_1 + \mathrm{o}(h), \tag{2.1}$$

where Δ is the co-multiplication in the universal enveloping algebra $U(\mathfrak{g})$. One can prove [8] that the map $\Delta_1 : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is such that $\Delta_1 - \sigma\Delta_1 = \delta$ (σ is the usual permutation) being restricted to $\mathfrak{g}$ gives a map $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ which is a 1-cocycle and defines the structure of a Lie coalgebra on $\mathfrak{g}$ (the structure of a Lie algebra on the dual space $\mathfrak{g}^*$). The pair $(\mathfrak{g}, \delta)$ is called a quasiclassical limit of $U_h(\mathfrak{g})$.

In general, a pair $(\mathfrak{g}, \delta)$, where $\mathfrak{g}$ is a Lie algebra and δ is such a 1-cocycle, is called a Lie bialgebra. It is proven [11] that any Lie bialgebra $(\mathfrak{g}, \delta)$ can be quantized, i.e., there exists a quantum group $U_h(\mathfrak{g})$ such that the pair $(\mathfrak{g}, \delta)$ is its quasiclassical limit.

A Lie bialgebra $(\mathfrak{g}, \delta)$ is said to be a coboundary one if there exists an element $r \in \wedge^2\mathfrak{g}$, called the classical r-matrix, such that $\delta(x) = [r, \Delta(x)]$ for $x \in \mathfrak{g}$. Since δ defines a Lie coalgebra structure, r has to satisfy the so-called classical Yang–Baxter equation which can be written in the form

$$[\![r, r]\!] = \varphi, \tag{2.2}$$

where $[\![\cdot, \cdot]\!]$ stands for the Schouten bracket and $\varphi \in \wedge^3\mathfrak{g}$ is an invariant element. We denote the coboundary Lie bialgebra by $(\mathfrak{g}, r)$.

In case $\mathfrak{g}$ is a simple Lie algebra, the most known r-matrix is the Sklyanin–Drinfeld one:

$$r = \sum_{\alpha} X_\alpha \wedge X_{-\alpha},$$

where the sum runs over all positive roots; the root vectors X_α are chosen in such a way that $(X_\alpha, X_{-\alpha}) = 1$ for the Killing form $(\cdot, \cdot)$. This is the only r-matrix of weight zero [18], and its quantization is the Drinfeld–Jimbo quantum group. A classification of all r-matrices for simple Lie algebras was given in [1].

From results of Drinfeld, and of Etingof and Kazhdan one can derive the following proposition.

Proposition 2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then*

1. *any Lie bialgebra $(\mathfrak{g}, \delta)$ is a coboundary one*;
2. *the quantization, $U_h(\mathfrak{g})$, of any coboundary Lie bialgebra $(\mathfrak{g}, r)$ exists and is isomorphic to $U(\mathfrak{g})[[h]]$ as a topological $\mathbb{C}[[h]]$ algebra*;
3. *the co-multiplication in $U_h(\mathfrak{g})$ has the form*

$$\Delta_h(x) = F_h \Delta(x) F_h^{-1}, \quad x \in U(\mathfrak{g}), \tag{2.3}$$

where $F_h \in U(\mathfrak{g})^{\otimes 2}[[h]]$ and can be chosen in the form

$$F_h = 1 \otimes 1 + \left(\frac{h}{2}\right) r + \mathrm{o}(h). \tag{2.4}$$

Proof. (1) follows from the fact that $H^1(\mathfrak{g}, \wedge^2\mathfrak{g}) = 0$. It follows from $H^2(\mathfrak{g}, U(\mathfrak{g})) = 0$ that $U(\mathfrak{g})$ does not admit any non-trivial deformations as an algebra (see [7]), which proves (2). From the fact that $H^1(\mathfrak{g}, U(\mathfrak{g})^{\otimes 2}) = 0$ it follows that any deformation of the algebra morphism $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ appears as a conjugation of Δ. In particular, the co-multiplication in $U_h(\mathfrak{g})$ looks like (2.3) with some F_h such that $F_0 = 1 \otimes 1$. It follows from the co-associativity of Δ_h that F_h satisfies the equation

$$(F_h \otimes 1) \cdot (\Delta \otimes \mathrm{id})(F_h) = (1 \otimes F_h) \cdot (\mathrm{id} \otimes \Delta)(F_h) \cdot \Phi_h \tag{2.5}$$

for some invariant element $\Phi_h \in U(\mathfrak{g})^{\otimes 3}[[h]]$.

The element F_h satisfying (2.3) and (2.4) can be obtained by a correction of some F_h only obeying (2.5) [8]. This procedure also uses a simple cohomological argument, which proves (3). □

It follows from (2.5) that if F_h has the form (2.4), then the coefficient by h in Φ_h vanishes. Moreover, the coefficient by h^2 is the element φ from (2.2), i.e.,

$$\Phi_h = 1 \otimes 1 \otimes 1 + h^2\varphi + \mathrm{o}(h^2). \tag{2.6}$$

In addition, it follows from (2.5) that Φ_h satisfies the pentagon identity

$$(\mathrm{id}^{\otimes 2} \otimes \Delta)(\Phi_h) \cdot (\Delta \otimes \mathrm{id}^{\otimes 2})(\Phi_h) = (1 \otimes \Phi_h) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi_h) \cdot (\Phi_h \otimes 1).$$

2.2. Equivariant deformation quantization

Let G be a simple connected complex Lie group whose Lie algebra is $\mathfrak{g}$. Let G act on a manifold M and $\mathcal{A}$ be the sheaf of functions on M. It may be the sheaf of analytic, smooth, or algebraic functions, dependingly of the type of M. Then $U(\mathfrak{g})$ acts on sections of $\mathcal{A}$, and the multiplication in $\mathcal{A}$ is $U(\mathfrak{g})$ invariant.

The deformation quantization of $\mathcal{A}$ is a sheaf of associative algebras, $\mathcal{A}_h$, which is isomorphic to $\mathcal{A}[[h]] = \mathcal{A} \otimes \mathbb{C}[[h]]$ (completed tensor product) as a $\mathbb{C}[[h]]$ module, with multiplication in $\mathcal{A}_h$ having the form $m_h = \sum_{k=0}^{\infty} h^k m_k$, where m_0 is the usual commutative multiplication in $\mathcal{A}$ and $m_k,\ k > 0$, are bidifferential operators vanishing on constants. The algebra $U(\mathfrak{g})[[h]]$ is clearly acts on the $\mathbb{C}[[h]]$ module $\mathcal{A}_h$.

We will study quantizations of $\mathcal{A}$ which are invariant under the $U_h(\mathfrak{g})$ action, i.e., under the co-multiplication Δ_h. This means that

$$bm_h(x \otimes y) = m_h\Delta_h(b)(x \otimes y) \quad \text{for } b \in U(\mathfrak{g}),\ x, y \in \mathcal{A}.$$

A $\mathbb{C}[[h]]$ linear map $\mu_h : \mathcal{A}_h \otimes \mathcal{A}_h \to \mathcal{A}_h$ is called a Φ_h *associative* multiplication if

$$\mu_h(\Phi_1 x \otimes \mu_h(\Phi_2 y \otimes \Phi_3 z)) = \mu_h(\mu_h(x \otimes y) \otimes z) \quad \text{for } x, y, z \in \mathcal{A},$$

where $\Phi_h = \Phi_1 \otimes \Phi_2 \otimes \Phi_3$ (summation implicit).

We say that the Φ_h associative multiplication $\mu_h = \sum_{k=0}^{\infty} h^k \mu_k$ gives a Φ_h associative quantization of $\mathcal{A}$ if $\mu_0 = m_0$, the usual multiplication in $\mathcal{A}$, and $\mu_k,\ k > 0$, are bidifferential operators vanishing on constants.

Proposition 2.2. *There is a natural one-to-one correspondence between $U_h(\mathfrak{g})$ invariant and $U(\mathfrak{g})$ invariant Φ_h associative quantizations of $\mathcal{A}$. Namely, if μ_h is a $U(\mathfrak{g})$ invariant Φ_h associative multiplication in $\mathcal{A}[[h]]$, then*

$$m_h = \mu_h F_h^{-1} \tag{2.7}$$

gives a $U_h(\mathfrak{g})$ invariant associative multiplication in $\mathcal{A}[[h]]$.

Proof. This follows immediately from (2.3) and (2.5). This follows also from the categorical interpretation of Φ_h and F_h [4,8]. □

This proposition shows that given a $U(\mathfrak{g})$ invariant Φ_h associative quantization of $\mathcal{A}$, we can get the $U_h(\mathfrak{g})$ invariant quantization of $\mathcal{A}$ for any quantum group $U_h(\mathfrak{g})$ from Proposition 2.1(2) by applying F_h from (2.4) to the Φ_h associative multiplication.

2.3. Poisson brackets associated with equivariant quantizations

A skew-symmetric map $f : \mathcal{A}^{\otimes 2} \to \mathcal{A}$ we call a *bracket* if it satisfies the Leibniz rule: $f(ab, c) = af(b, c) + f(a, c)b$ for $a, b, c \in \mathcal{A}$. It is easy to see that any bracket is presented by a bivector field on M. Further, we will identify brackets and bivector fields on M.

For an element $\psi \in \wedge^k \mathfrak{g}$ we denote by ψ_M the k-vector field on M which is induced by the action map $\mathfrak{g} \to \mathrm{Vect}(M)$. A bracket f is a Poisson one if the Schouten bracket $[\![f, f]\!]$ is equal to zero.

Definition 2.1. A G invariant bracket f on M we call a φ-bracket if $[\![f, f]\!] = -\varphi_M$, where $\varphi \in \wedge^3 \mathfrak{g}$ is an invariant element.

Proposition 2.3. *Let $\mathcal{A}_h$ be a $U(\mathfrak{g})$ invariant Φ_h associative quantization with multiplication $\mu_h = m_0 + h\mu_1 + \mathrm{o}(h)$, where m_0 is the multiplication in $\mathcal{A}$. Then the map $f : \mathcal{A}^{\otimes 2} \to \mathcal{A}$, $f(a, b) = \mu_1(a, b) - \mu_1(b, a)$, is a φ-bracket for φ from* (2.6).

Proof. A direct computation. Another proof is found in [4]. □

Corollary 2.1. *Let $\mathcal{A}_h$ be a $U_h(\mathfrak{g})$ invariant associative quantization with multiplication $m_h = m_0 + hm_1 + \mathrm{o}(h)$. Then the corresponding Poisson bracket $p(a, b) = m_1(a, b) - m_1(b, a)$ has the form*

$$p(a, b) = f(a, b) - r_M(a, b), \tag{2.8}$$

where r is the r-matrix corresponding to $U_h(\mathfrak{g})$ and f is a φ-bracket with $\varphi = [\![r, r]\!]$.

Proof. By Proposition 2.2 there is a $U(\mathfrak{g})$ invariant Φ_h associative multiplication μ_h such that $m_h = \mu_h F_h^{-1}$ with F_h as in (2.4). Let f be the φ-bracket corresponding to μ_h. Then a direct computation shows that the Poisson bracket of m_h is as required. □

Remark 2.1. For an r-matrix $r \in \wedge^2 \mathfrak{g}$, denote by r' and r'' the left and right invariant bivector fields on G corresponding to r. Then it follows from (2.2) that the bivector field $r' - r''$ defines a Poisson bracket on G which makes G into a Poisson–Lie group. On the other hand, a Poisson brackets on M admitting, in principle, a $U_h(\mathfrak{g})$ invariant quantization endows M with a structure of (G, r)-manifold. This means that the action $G \times M \to M$ is a Poisson map. So, Corollary 2.1 describes the form of Poisson brackets on M making M into a (G, r)-manifold. One sees, in particular, that the classification of (G, r) Poisson structures on M reduces to the classification of φ-brackets on M.

We will also consider two parameter quantizations on M. A two parameter quantization of $\mathcal{A}$ is an algebra $\mathcal{A}_{t,h}$ isomorphic to $\mathcal{A}[[t,h]]$ as a $\mathbb{C}[[t,h]]$ module and having a multiplication of the form

$$m_{t,h} = m_0 + tm_1' + hm_1'' + \mathrm{o}(t,h). \tag{2.9}$$

With such a quantization one associates two Poisson brackets: the bracket $v(a,b) = m_1'(a,b) - m_1'(b,a)$ along t, and the bracket $p(a,b) = m_1''(a,b) - m_1''(b,a)$ along h. It is easy to check that p and v are compatible Poisson brackets, i.e., their Schouten bracket $[\![p,v]\!]$ is equal to zero.

Corollary 2.2. *Let $\mathcal{A}_{t,h}$ be a $U_h(\mathfrak{g})$ invariant associative quantization of the form* (2.9). *Then the Poisson bracket $p(a,b) = m_1''(a,b) - m_1''(b,a)$ has the form*

$$p(a,b) = f(a,b) - r_M(a,b),$$

where r is the r-matrix corresponding to $U_h(\mathfrak{g})$ and f is a φ-bracket with $\varphi = [\![r,r]\!]$. The Poisson bracket $v(a,b) = m_1'(a,b) - m_1'(b,a)$ is invariant and compatible with p.

Proof. Similar to Corollary 2.1. □

In the following, a φ-bracket on M compatible with a non-degenerate Poisson bracket we call a *good* bracket.

3. Classification of φ- and good brackets on semisimple orbits

Let G be a complex connected simple Lie group with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$, the Levi factor of a parabolic subalgebra. Let L be a Lie subgroup of G with Lie algebra $\mathfrak{l}$. Such a subgroup is called a Levi subgroup. It is known that L is a closed connected subgroup. Denote $M = G/L$ and let $o \in M$ be the image of the unity by the natural projection $G \to M$. Then L is the stabilizer of o. It is known that M may be realized as a semisimple orbit of G in the coadjoint representation $\mathfrak{g}^*$. Conversely, any semisimple orbit in $\mathfrak{g}^*$ is a quotient of G by a Levi subgroup.

Let $\mathfrak{h} \subset \mathfrak{l}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\Omega_{\mathfrak{l}} \subset \Omega \subset \mathfrak{h}^*$ the sets of roots of $\mathfrak{l}$ and $\mathfrak{g}$ corresponding to $\mathfrak{h}$. Choose root vectors $E_\alpha,\ \alpha \in \Omega$, in such a way that

$$(E_\alpha, E_{-\alpha}) = 1 \tag{3.1}$$

for the Killing form $(\cdot,\cdot)$ on $\mathfrak{g}$.

Let Q be a set embedded in a linear space V such that $0 \notin Q$ and $Q = -Q$. We call a subset $B \subset Q$ a *linear subset* if $B = Q \cap V_B$ where V_B is the linear subspace in V generated by B. We call a subset $B \subset Q$ *semi-linear* if it follows from $x, y \in B,\ x + y \in Q$ that $x + y \in B$, and, in addition, $B \cap (-B) = \varnothing,\ B \cup (-B) = Q$. For a linear subset B of Q, we denote by Q/B the image of Q without zero by the projection $F \to F/V_B$.

Since $\mathfrak{l}$ is a Levi subalgebra, $\Omega_{\mathfrak{l}}$ is a linear subset in Ω. We put $\bar{\Omega} = \Omega/\Omega_{\mathfrak{l}}$ and call elements of $\bar{\Omega}$ *quasiroots*. For $\alpha \in \Omega$ we denote by $\bar{\alpha}$ its image in $\bar{\Omega}$. Let Y be a semi-linear subset in $\bar{\Omega}$. One can easily shown that there is a subset $P \subset Y$ such that any element of Y can be uniquely presented as a linear combination of elements of P with integer coefficients. We call P a *set of simple quasiroots* corresponding to Y and Y *a set of positive quasiroots* with respect to P. It is clear that there is a set of simple roots, Π, in Ω such that $P = \bar{\Pi}$. Then $Y = \bar{\Omega}^+$, where Ω^+ is the system of positive roots corresponding to Π. For such a Π there is a subset, $\Gamma \subset \Pi$, such that $\mathfrak{l}$ coincides with the Lie subalgebra $\mathfrak{g}_\Gamma$, the subalgebra generated by $\mathfrak{h}$ and elements $E_{\pm\alpha}$, $\alpha \in \Gamma$.

The projection $\pi : G \to M$ induces the map $\pi_* : \mathfrak{g} \to T_{\mathrm{o}}$ where T_{o} is the tangent space to M at the point o. Since the ad-action of $\mathfrak{l}$ on $\mathfrak{g}$ is semisimple, there exists an $\mathrm{ad}(\mathfrak{l})$-invariant subspace $\mathfrak{m} = \mathfrak{m}_{\mathfrak{l}}$ of $\mathfrak{g}$ complementary to $\mathfrak{l}$, and one can identify T_{o} and $\mathfrak{m}$ by means of π_*. It is easy to see that subspace $\mathfrak{m}$ is uniquely defined and has a basis consisting of the elements E_γ, $\gamma \in \Omega \setminus \Omega_{\mathfrak{l}}$.

Proposition 3.1. *The space* $\mathfrak{m}$ *considered as a* $\mathfrak{l}$ *representation space decomposes into the direct sum of sub-representations* $\mathfrak{m}_{\bar{\beta}}$, $\bar{\beta} \in \bar{\Omega}$, *where* $\mathfrak{m}_{\bar{\beta}}$ *is generated by all the elements* E_β, $\beta \in \Omega$, *such that the projection of* β *is equal to* $\bar{\beta}$. *This decomposition have the following properties*:

1. *all* $\mathfrak{m}_{\bar{\beta}}$ *are irreducible*;
2. $\mathfrak{m}_{-\bar{\beta}}$ *is dual to* $\mathfrak{m}_{\bar{\beta}}$;
3. *for* $\bar{\beta}_1, \bar{\beta}_2 \in \bar{\Omega}$ *such that* $\bar{\beta}_1 + \bar{\beta}_2 \in \bar{\Omega}$ *one has* $[\mathfrak{m}_{\bar{\beta}_1}, \mathfrak{m}_{\bar{\beta}_2}] = \mathfrak{m}_{\bar{\beta}_1+\bar{\beta}_2}$;
4. *for any pair* $\bar{\beta}_1, \bar{\beta}_2 \in \bar{\Omega}$ *the representation* $\mathfrak{m}_{\bar{\beta}_1} \otimes \mathfrak{m}_{\bar{\beta}_2}$ *is multiplicity-free*.

Proof. Statements (1)–(3) are proven in [4, Remark 3.1]. Statement (4) follows from the fact that the weight subspaces of all $\mathfrak{m}_{\bar{\beta}}$ have dimension 1 (see N. Bourbaki, Groupes et algébres de Lie, Chapter 8, Section 9, Ex. 14). □

Restricting to the point $o \in M$ defines the natural one-to-one correspondence between G invariant tensor fields on M and $\mathfrak{l}$ invariant tensors over $\mathfrak{m}$.

Since $\mathfrak{l}$ contains a Cartan subalgebra, $\mathfrak{h}$, each $\mathfrak{l}$ invariant tensor over $\mathfrak{m}$ is of weight zero with respect to $\mathfrak{h}$. It follows that there are no invariant vectors in $\mathfrak{m}$. Hence, there are no invariant vector fields on M.

Proposition 3.2. *A bivector* $v \in \wedge^2\mathfrak{m}$ *is* $\mathfrak{l}$ *invariant if and only if it has the form* $v = \frac{1}{2}\sum c(\bar{\alpha})E_\alpha \wedge E_{-\alpha}$, *where the sum runs over* $\alpha \in \Omega \setminus \Omega_{\mathfrak{l}}$ (*we suppose* $c(-\bar{\alpha}) = -c(\bar{\alpha})$).

Proof. Follows from (3.1) and Proposition 3.1 (see also [4, Proposition 3.2]). □

Denote by $[\![v, w]\!] \in \wedge^{k+l-1}\mathfrak{m}$ the Schouten bracket of polyvector fields $v \in \wedge^k\mathfrak{m}$ and $w \in \wedge^l\mathfrak{m}$ defined by the formula

$$lobrk[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l]\!] = \sum (-1)^{i+j}[X_i, Y_j]_{\mathfrak{m}} \wedge X_1 \wedge \cdots \widehat{X}_i \cdots \widehat{Y}_j \cdots \wedge Y_l,$$

where $[\cdot,\cdot]_{\mathfrak{m}}$ is the composition of the Lie bracket in $\mathfrak{g}$ and the projection $\mathfrak{g}\to\mathfrak{m}$. The defined Schouten bracket is compatible with the Schouten bracket on M under identifying $\mathfrak{l}$ invariant polyvectors over $\mathfrak{m}$ and G invariant polyvector fields on M.

It is obvious that any $\mathfrak{l}$ invariant bivector is θ anti-invariant for the Cartan automorphism θ, $\theta(E_\alpha)=-E_{-\alpha}$, of $\mathfrak{g}$. Hence, if $v,w\in\wedge^2\mathfrak{m}$ are $\mathfrak{l}$ invariant, then $[\![v,w]\!]$ is θ invariant, i.e., is of the form $[\![v,w]\!]=\sum e(\alpha,\beta)E_{\alpha+\beta}\wedge E_{-\alpha}\wedge E_{-\beta}$, where $e(\alpha,\beta)=-e(-\alpha,-\beta)$. Hence, in order to calculate $[\![v,w]\!]$ for such v and w it is sufficient to calculate coefficients $e(\alpha,\beta)$ for positive α and β by any choice of the system of positive roots.

Lemma 3.1. *Let $v=\sum c(\alpha)E_\alpha\wedge E_{-\alpha}$, $w=\sum d(\alpha)E_\alpha\wedge E_{-\alpha}$ be elements from $\mathfrak{g}\wedge\mathfrak{g}$. Choose a system of positive roots. Then for any positive roots $\alpha,\beta,(\alpha+\beta)$ the coefficient by the term $E_{\alpha+\beta}\wedge E_{-\alpha}\wedge E_{-\beta}$ in $[\![v,w]\!]$ is equal to*

$$N_{\alpha,\beta}(d(\alpha)(c(\beta)-c(\alpha+\beta))+d(\beta)(c(\alpha)-c(\alpha+\beta))-d(\alpha+\beta)(c(\alpha)+c(\beta))), \tag{3.2}$$

where the number $N_{\alpha,\beta}$ is defined by relation $[E_\alpha,E_\beta]=N_{\alpha,\beta}E_{\alpha+\beta}$.

Proof. Direct computation (see [16]). □

Let $\varphi\in\wedge^2\mathfrak{g}$ be an invariant element. Since $\mathfrak{g}$ is simple, φ is defined uniquely up to a factor. Denote by φ_M the invariant three-vector field on M induced by φ with the help of the action map $\mathfrak{g}\to\mathrm{Vect}(M)$. It is easy to check that φ_M is θ invariant and up to a factor has the form

$$\varphi_M=\frac{1}{3}\sum_{\alpha,\beta,\alpha+\beta\in\Omega\setminus\Omega_1}N_{\alpha,\beta}E_{\alpha+\beta}\wedge E_{-\alpha}\wedge E_{-\beta}. \tag{3.3}$$

From Lemma 3.1, it follows that the Schouten bracket of bivector $v=\frac{1}{2}\sum c(\bar\alpha)E_\alpha\wedge E_{-\alpha}$ with itself is equal to $K^2\varphi_M$ for a complex number K, if and only if the following equation holds:

$$c(\bar\alpha+\bar\beta)(c(\bar\alpha)+c(\bar\beta))=c(\bar\alpha)c(\bar\beta)+K^2 \tag{3.4}$$

for all the pairs of quasiroots $\bar\alpha,\bar\beta$ such that $\bar\alpha+\bar\beta$ is a quasiroot.

Let X_{K^2} be the algebraic variety consisting of the points $\{c(\bar\alpha),\bar\alpha\in\bar\Omega\}$ satisfying (3.4) (we always assume $c(\bar\alpha)=-c(-\bar\alpha)$). So, for a given φ the variety $X=X_{K^2=-1}$ is the variety of all φ-brackets. It is clear that all the varieties X_{K^2}, $K\neq 0$, are isomorphic to X.

Let $\{c(\bar\alpha)\}$ be a solution of (3.4) for a number K, i.e., $\{c(\bar\alpha)\}\in X_{K^2}$. It is easy to derive the following properties: $(*)$ If $c(\bar\alpha)+c(\bar\beta)=0$ then necessarily $c(\bar\alpha)=\pm K$, $c(\bar\beta)=\mp K$. $(**)$ If $c(\bar\alpha)=\pm K$ and $c(\bar\beta)\neq\pm K$, then $c(\bar\alpha+\bar\beta)=\pm K$ and $c(\bar\alpha-\bar\beta)=\pm K$. $(***)$ If $c(\bar\alpha)=\pm K$ and $c(\bar\beta)=\pm K$, then $c(\bar\alpha+\bar\beta)=\pm K$.

Formally, all the solutions of (3.4) for a fixed K can be obtained in the following way. Choose a system of positive quasiroots, $\bar{\Omega}^+$. Denote by $\bar{\Pi}$ the corresponding set of simple quasiroots. Given $c(\bar{\alpha})$ and $c(\bar{\beta})$, we find from (3.4) that

$$c(\bar{\alpha}+\bar{\beta}) = \frac{c(\bar{\alpha})c(\bar{\beta})+K^2}{c(\bar{\alpha})+c(\bar{\beta})}. \tag{3.5}$$

Assume, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are positive quasiroots such that $\bar{\alpha}+\bar{\beta}$, $\bar{\beta}+\bar{\gamma}$, $\bar{\alpha}+\bar{\beta}+\bar{\gamma}$ are also quasiroots. Then the number $c(\bar{\alpha}+\bar{\beta}+\bar{\gamma})$ can be calculated formally (ignoring possible division by zero) in two ways, using (3.5) for the pair $c(\bar{\alpha})$, $c(\bar{\beta}+\bar{\gamma})$ on the right-hand side and also for the pair $c(\bar{\alpha}+\bar{\beta})$, $c(\bar{\gamma})$. But it is easy to check that these two ways give the same value of $c(\bar{\alpha}+\bar{\beta}+\bar{\gamma})$. In this sense the system of equations corresponding to (3.5) for all pairs is consistent. So, taking arbitrary values $c(\bar{\alpha})$ for simple quasiroots $\bar{\alpha}$ one can try to find $c(\bar{\alpha})$ for all $\bar{\alpha}\in\bar{\Omega}^+$ recursively. We say that a solution, $\{c(\bar{\alpha})\}$, of (3.4) can be obtained recursively if in the course of the recursive procedure started with the values $c(\bar{\alpha})$ for simple quasiroots $\bar{\alpha}$ the denominators in (3.5) will be not equal to zero.

Proposition 3.3. *For $K\neq 0$, the following holds*:

1. *Any solution of* (3.4) *can be obtained recursively by choosing a respective system of positive quasiroots*.
2. *The variety X_{K^2} is without singularities, connected, and of dimension k, where k is equal to the number of simple quasiroots*.

Proof. For proving (1) we have to show that for any solution $\{c(\bar{\alpha})\}\in X_{K^2}$ one can choose a system of positive quasiroots in such a way that the denominators appearing in (3.5) by the recursive procedure are not equal to zero. It follows from $(**)$ that the set Ψ consisting of $\bar{\alpha}$ such that $c(\bar{\alpha})\neq\pm K$ is a linear subset of $\bar{\Omega}$. Moreover, the function $c(\bar{\alpha})$ is constant on the cosets of $\bar{\Omega}/\Psi$. Let Y is the set of cosets on which this function has the value K. It follows from $(***)$ that Y is a semi-linear subset of $\bar{\Omega}/\Psi$. Let $\bar{\Omega}^+$ be a semi-linear subset of $\bar{\Omega}$ projecting on Y. Then it follows from $(*)$ that for $\bar{\alpha},\bar{\beta}\in\bar{\Omega}^+$, $c(\bar{\alpha})+c(\bar{\beta})\neq 0$, which proves (1).

Let $\bar{\Pi}=\{\bar{\alpha}_i,\ i=1,\dots,k\}$ be the set of simple quasiroots corresponding to $\bar{\Omega}^+$. Let $c(\bar{\alpha}_i)=c_i$. It is clear that starting the recursive procedure with $c(\bar{\alpha}_i)=c'_i$ for c'_i arbitrary but close enough to c_i, the denominators in (3.5) remain not equal to zero. This proves that any point of X_{K^2} is non-singular and X_{K^2} has dimension k.

Let us prove that X_{K^2} is connected. Fix a set of positive quasiroots, $\bar{\Omega}^+$, and the corresponding set of simple quasiroots, $\bar{\Pi}=\{\bar{\alpha}_i,\ i=1,\dots,k\}$. We say that a k-tuple of complex numbers $(c_1,\dots,c_k)$ is admissible, if starting with $c(\bar{\alpha}_i)=c_i$ one obtains a solution of (3.4) by the recursive procedure. It is clear that the admissible tuples form a subset, A, of $\mathbb{C}^k$ complement to an algebraic subset of lesser dimension, therefore A is connected. On the other hand, the set of points $\{c(\bar{\alpha})\}\in X_{K^2}$ such that $c(\bar{\alpha}_i)$ form an admissible k-tuple is obviously dense in X_{K^2}. This proves the correctness of X_{K^2}. □

Let $\{c(\bar{\alpha})\}$ be a solution of (3.4) and $\Psi\in\bar{\Omega}$ the linear subset of $\bar{\Omega}$ such that $c(\bar{\alpha})\neq\pm K$ for $\bar{\alpha}\in\Psi$. Then, using the formula for $\coth(x+y)$, similar to (3.5), one can see that there exists a linear form $\lambda : \Psi \to \mathbb{C}$ such that

$$\lambda(\bar{\alpha}) \notin \frac{2\pi \mathrm{i}}{K}\mathbb{Z}, \quad \bar{\alpha} \in \Psi, \tag{3.6}$$

$$c(\bar{\alpha}) = K \coth(\tfrac{1}{2}K\lambda(\bar{\alpha})) \quad \text{for any } \bar{\alpha} \in \Psi. \tag{3.7}$$

As $K \to 0$, (3.6) makes into

$$\lambda(\bar{\alpha}) \neq 0, \quad \bar{\alpha} \in \Psi, \tag{3.8}$$

and (3.7) tends to

$$c(\bar{\alpha}) = \frac{1}{\lambda(\bar{\alpha})} \quad \text{for any } \bar{\alpha} \in \Psi. \tag{3.9}$$

So, we come to the following proposition.

Proposition 3.4.

1. *For $K \neq 0$, any solution of* (3.4) *is determined by: choosing a linear subset, Ψ, in $\bar{\Omega}$, a semi-linear subset, B, in $\bar{\Omega}/\Psi$, and a linear form, $\lambda : \Psi \to \mathbb{C}$, satisfying* (3.6). *The respective solution $\{c(\bar{\alpha})\}$ is the following: for $\bar{\alpha} \in \Psi$, $c(\bar{\alpha})$ is defined by* (3.7); *for $\bar{\alpha} \notin \Psi$, $c(\bar{\alpha}) = K$ if the projection of $\bar{\alpha}$ in $\bar{\Omega}/\Psi$ belongs to B, $c(\bar{\alpha}) = -K$ if the projection belongs to $-B$.*
2. *For $K = 0$, any solution of* (3.4) *is defined by: choosing a linear subset, Ψ, in $\bar{\Omega}$ and a linear form, $\lambda : \Psi \to \mathbb{C}$, satisfying* (3.8). *The solution $\{c(\bar{\alpha})\}$ is the following: for $\bar{\alpha} \in \Psi$, $c(\bar{\alpha})$ is defined by* (3.9); *for $\bar{\alpha} \notin \Psi$, $c(\bar{\alpha}) = 0$.*

Remark 3.1. As mentioned in [17], for $\mathfrak{l} = \mathfrak{h}$ the solutions described in Proposition 3.4 relate to solutions of classical dynamical Yang–Baxter equations [12].

Note that by $K = 0$ the solutions of (3.4) define Poisson brackets on M, so Proposition 3.4(2) describes all the Poisson brackets on M. We see that non-degenerate Poisson brackets on M are in one-to-one correspondence with the linear forms $\lambda : \bar{\Omega} \to \mathbb{C}$ such that $\lambda(\bar{\alpha}) \neq 0$ for all $\bar{\alpha} \in \bar{\Omega}$ and have the form

$$\frac{1}{2}\sum_{\bar{\alpha}\in\bar{\Omega}} \frac{1}{\lambda(\bar{\alpha})} E_\alpha \wedge E_{-\alpha}. \tag{3.10}$$

This is exactly the KKS bracket on the orbit in $\mathfrak{g}^*$ passing through the linear form on $\mathfrak{g}$ being the trivial extension of λ.

Denote by X_0 the variety of non-degenerate Poisson brackets on M. Since X_0 coincides with all solutions of (3.4), $\{c(\bar{\alpha})\}$, such that $\Pi_{\bar{\alpha}} c(\bar{\alpha}) \neq 0$, it is clear that X_0 is an affine connected algebraic variety without singularities.

Fix a Poisson bracket s of the form (3.10). Let us describe the invariant brackets $f = \sum c(\bar{\alpha}) E_\alpha \wedge E_{-\alpha}$ satisfying the conditions

$$[\![f, f]\!] = K^2 \varphi_M, \tag{3.11}$$

$$[\![f, s]\!] = 0 \tag{3.12}$$

with some $K \neq 0$.

A direct computation shows that conditions (3.11) and (3.12) are equivalent to the system of equations for the coefficients $c(\bar{\alpha})$ of f [4],

$$c(\bar{\beta})\lambda(\bar{\beta}) = c(\bar{\alpha})\lambda(\bar{\alpha}) \pm K\lambda(\bar{\alpha} + \bar{\beta}), \tag{3.13}$$

$$c(\bar{\alpha} + \bar{\beta})\lambda(\bar{\alpha} + \bar{\beta}) = c(\bar{\alpha})\lambda(\bar{\alpha}) \pm K\lambda(\bar{\beta}) \tag{3.14}$$

with the same sign before K, for all the pairs of quasiroots $\bar{\alpha}$, $\bar{\beta}$ such that $\bar{\alpha}+\bar{\beta}$ is a quasiroot.

Definition 3.1. Let M be an orbit in $\mathfrak{g}^*$ (not necessarily semisimple). The invariant bracket f on M is said to be *good* if f satisfies conditions (3.11) and (3.12) for s the Kirillov–Kostant–Souriau (KKS) Poisson bracket on M. We call M a *good* orbit, if there exists a good bracket on it.

Proposition 3.5.

1. *For* $\mathfrak{g}$ *of type* A_n *all semisimple orbits are good*.
2. *For all other* $\mathfrak{g}$, *the orbit M is good if and only if* $\mathfrak{l} = \mathfrak{g}_\Gamma$, *where* $\Gamma \subset \Pi$ *for a system* Π *of simple root for* $\mathfrak{g}$ *and the set* $\Pi \setminus \Gamma$ *consists of one or two roots which appear in the representation of the maximal root with coefficient* 1.
3. *For a given* $K \neq 0$ *the good brackets f on a good orbit form a one-dimensional variety: all such brackets have the form*

   $$\pm f_0 + ts,$$

   where $t \in \mathbb{C}$ *and* f_0 *is a fixed bracket satisfying* (3.11) *and* (3.12).

Proof. The proof reduces to solving the system of equations defined by (3.11) and (3.12) (see [4]). □

So, if the set $\Pi \setminus \Gamma$ consists of one root, M is exactly a Hermitian symmetric space. As follows from the classification of simple Lie algebras, the case when the set $\Pi \setminus \Gamma$ consists of two roots appears (besides A_n) for $\mathfrak{g}$ of types D_n and E_6.

Proposition 3.5 shows that the property for M to be a good orbit depends only on the pair $(\mathfrak{g}, \mathfrak{l})$ but not on the realization of M as an orbits. The pair $(\mathfrak{g}, \mathfrak{l})$ corresponding to a good orbit we call a good pair.

Remark 3.2. It is clear that if f satisfies (3.11) and (3.12) then $\pm f + ts$ also satisfies the same conditions (with the same K) for all numbers t. Proposition 3.5(3) shows that, conversely, all good brackets on a good semisimple orbit are contained in these families $\pm f + ts$, $t \in \mathbb{C}$.

Denote by Y the variety of good brackets f on M satisfying (3.11) and (3.12) for a fixed $K \neq 0$ and some non-degenerate Poisson bracket s. From the above, it follows that there is a projection, $Y \to X_0,\ f \mapsto s$, where s is a Poisson bracket such that $[\![f, s]\!] = 0$. The fiber over $s \in X_0$ consists of two components, $\{\pm f_0 + ts\},\ t \in \mathbb{C}$, isomorphic to $\mathbb{C}$. These components correspond to choosing the sign in (3.13) and (3.14).

Remark 3.3. It is shown in [5] that in case A_n, i.e., when $\mathfrak{g} = \mathrm{sl}(n)$, all the coadjoint orbits (not necessarily semisimple) are good. Moreover, there exists a unique quadratic φ-bracket on $\mathrm{sl}(n)^*$ which can be restricted to all orbits to give good brackets on them. This quadratic bracket on $\mathrm{sl}(n)^*$ can be quantized [6].

4. Poisson complexes

Let $C^k = (\Lambda^k\mathfrak{m})^{\mathfrak{l}}$ be the space of $\mathfrak{l}$ invariant k-vectors on $\mathfrak{m}$. This space is identified with the space of G invariant k-vector fields on M. Denote by $\mathcal{C}^k$ the sheaf of holomorphic functions on X_{K^2} with values in C^k. We form the complex $(C^\bullet, \delta_f)$ where δ_f is the differential given by the Schouten bracket with a bivector $f \in X_{K^2}$,

$$\delta_f : u \mapsto [\![f, u]\!] \quad \text{for } u \in C^\bullet.$$

The condition $\delta_f^2 = 0$ follows from the Jacobi identity for the Schouten bracket together with the fact that $[\![f_x, f_x]\!] = K^2\varphi_M$.

We also consider the complex of sheaves $(\mathcal{C}^\bullet, \delta)$ on X_{K^2}. The operator δ is defined as

$$\delta(u)(f) = [\![f, u(f)]\!] = \delta_f(u(f)), \tag{4.1}$$

where u is a section of $\mathcal{C}^\bullet$ and $f \in X_{K^2}$.

We denote by $H^k(M, \delta_f)$ and $H^k(\mathcal{C}^\bullet, \delta)$ the cohomologies of $(C^\bullet, \delta_f)$ and $(\mathcal{C}^\bullet, \delta)$, respectively, whereas the usual de Rham cohomologies are denoted by $H^k(M)$.

Proposition 4.1.

1. *For any non-degenerate Poisson brackets $f \in X_0$ and, if $K \neq 0$, for almost all $f \in X_{K^2}$* (*except an algebraic subset of lesser dimension*), *one has*
$$H^k(M, \delta_f) = H^k(M) \tag{4.2}$$
for all k. In particular, $H^k(M, \delta_f) = 0$ for odd k.
2. *Let $K \neq 0$. Then $H^2(M, \delta_f) = H^2(M)$ for all $f \in X_{K^2}$.*

Proof. The proof of (1) follows [4]. First, let v be a non-degenerate Poisson bracket on M, in particular, $v \in X_0$. Then the complex of polyvector fields on M, $\Theta^\bullet$, with the differential δ_v is well defined. Denote by $\Omega^\bullet$ the de Rham complex on M. Since none of the coefficients $c(\bar{\alpha})$ of v are zero, v is a non-degenerate bivector field, and therefore it defines an $\mathcal{A}$-linear isomorphism $\widetilde{v} : \Omega^1 \to \Theta^1,\ \omega \mapsto v(\omega, \cdot)$, which can be extended up to the isomorphism $\widetilde{v} : \Omega^k \to \Theta^k$ of k-forms onto k-vector fields for all k. Using Jacobi identity for v and invariance of v, one can show that $\widetilde{v}$ gives a G invariant isomorphism of these complexes, so their cohomologies are the same.

Since $\mathfrak{g}$ is simple, the subcomplex of $\mathfrak{g}$ invariants, $(\Omega^\bullet)^{\mathfrak{g}}$, splits off as a sub-complex of $\Omega^\bullet$. In addition, $\mathfrak{g}$ acts trivially on cohomologies, since for any $g \in G$ the map $M \to M,\ x \mapsto gx$, is homotopic to the identity map (G is a connected Lie group corresponding to $\mathfrak{g}$). It follows that cohomologies of complexes $(\Omega^\bullet)^{\mathfrak{g}}$ and $\Omega^\bullet$ coincide.

But $\widetilde{v}$ gives an isomorphism of complexes $(\Omega^\bullet)^{\mathfrak{l}}$ and $(\Theta^\bullet)^{\mathfrak{g}} = ((\Lambda^\bullet \mathfrak{m})^{\mathfrak{l}}, \delta_v)$. So, cohomologies of the latter complex coincide with de Rham cohomologies, which proves (1) for v being Poisson brackets.

Denote by X the variety of points $\{c(\bar\alpha), \bar\alpha \in \bar\Omega; K\}$ satisfying (3.4). Consider the family of complexes $((\Lambda^\bullet \mathfrak{m})^{\mathfrak{l}}, \delta_v),\ v \in X$. It is clear that δ_v depends algebraically on v. It follows from the upper semi-continuity of dim $H^k(M, \delta_v)$ and the fact that $H^k(M) = 0$ for odd k [2] that $H^k(M, \delta_v) = 0$ for odd k and almost all $v \in X$. Using the upper semi-continuity again and the fact that the number $\sum_k (-1)^k \dim H^k(M, \delta_v)$ is the same for all $v \in X$, we conclude that $\dim H^k(M, \delta_v) = \dim H^k(M)$ for even k and almost all $v \in X$. We have that there is a number $K_0 \neq 0$ and $v \in X_{K_0}$, such that (4.2) holds for δ_v. Since the cohomologies do not change when v replaces by cv with any number $c \neq 0$, we conclude that for any $K \neq 0$ there is f satisfying (4.2). Since by Proposition 3.3(2) X_{K^2} is connected, it follows that (4.2) holds for almost all $f \in X_{K^2}$.

Let us prove (2). Since $(\Lambda^1 \mathfrak{m})^{\mathfrak{l}} = 0,\ H^2(M, \delta_f)$ coincides with $\mathrm{Ker}(\delta_f : C^2 \to C^3)$. According to Proposition 3.3, let us choose a system of positive quasiroots $\bar\Omega^+$ such that f has the form $f = \sum_{\bar\alpha \in \bar\Omega^+} c(\bar\alpha) E_\alpha \wedge E_{-\alpha}$ and for $\bar\alpha, \bar\beta, \bar\alpha + \bar\beta \in \bar\Omega^+,\ c(\bar\alpha) + c(\bar\beta) \neq 0$. By Lemma 3.1, any $v \in C^2$ such that $[\![f, v]\!] = 0$ has the form $v = \sum_{\bar\alpha \in \bar\Omega^+} d(\bar\alpha) E_\alpha \wedge E_{-\alpha}$, where $d(\bar\alpha)$ obey the equation

$$d(\alpha)(c(\beta) - c(\alpha+\beta)) + d(\beta)(c(\alpha) - c(\alpha+\beta)) - d(\alpha+\beta)(c(\alpha) + c(\beta)) = 0. \tag{4.3}$$

Starting with arbitrary values $d(\bar\alpha)$ for simple quasiroots we find $d(\bar\alpha)$ for all $\bar\alpha \in \bar\Omega^+$ recursively using the formula following from (4.3):

$$d(\bar\alpha + \bar\beta) = \frac{d(\alpha)(c(\beta) - c(\alpha+\beta)) + d(\beta)(c(\alpha) - c(\alpha+\beta))}{c(\alpha) + c(\beta)}. \tag{4.4}$$

Indeed, denominators of (4.4) are not equal to zero by choosing of $\bar\Omega^+$. Moreover, assume $\bar\alpha, \bar\beta, \bar\gamma$ are positive quasiroots such that $\bar\alpha + \bar\beta, \bar\beta + \bar\gamma, \bar\alpha + \bar\beta + \bar\gamma$ are also quasiroots. Then the number $d(\bar\alpha + \bar\beta + \bar\gamma)$ can be calculated in two ways, using (4.4) for the pair $d(\bar\alpha), d(\bar\beta + \bar\gamma)$ on the right-hand side and also for the pair $d(\bar\alpha + \bar\beta), d(\bar\gamma)$. But it is easy to check that these two ways give the same value of $d(\bar\alpha + \bar\beta + \bar\gamma)$.

So we see that $\dim H^2(M, \delta_f) = \{$the number of simple quasiroots$\}$ for all $f \in X_{K^2}$. □

Proposition 4.2. *Let* $K \neq 0$. *Then*

$$H^3(\Gamma\mathcal{C}^\bullet, \delta) = 0,$$

where $\Gamma\mathcal{C}^k$ *is the space of global sections of* $\mathcal{C}^k$ *over* X_{K^2}.

Proof. Since X_{K^2} is a Stein manifold, it is enough to prove that $H^3(\mathcal{C}^\bullet, \delta) = 0$. Note that according to Proposition 4.1(2), $\mathrm{Ker}(\delta : \mathcal{C}^2 \to \mathcal{C}^3) = H^2(\mathcal{C}^\bullet, \delta)$ is a sub-bundle (direct sub-sheaf) of $\mathcal{C}^2$. Therefore, $\mathrm{Im}(\delta)$ is a sub-bundle of $\mathcal{C}^3$. On the other hand, $\mathrm{Ker}(\delta : \mathcal{C}^3 \to \mathcal{C}^4)$ being a sub-sheaf of $\mathcal{C}^3$ is a torsion-free sheaf. It follows that $H^3(\mathcal{C}^\bullet, \delta)$ is a torsion-free sheaf. According to Proposition 4.1(1), the support of $H^3(\mathcal{C}^\bullet, \delta)$ is an algebraic subset of X_{K^2} of lesser dimension. Hence, $H^3(\Gamma\mathcal{C}^\bullet, \delta) = 0$. □

Let M be a good orbit, s the KKS Poisson bracket on M, and f_0 a good bracket on M. Hence, $[\![f_0, f_0]\!] = K^2\varphi_M$ for some $K \neq 0$ and $[\![f_0, s]\!] = 0$. Consider the family of brackets

$$\phi = f_{h,t} = hf_0 + ts, \quad h, t \in \mathbb{C}.$$

One has $[\![f_{h,t}, f_{h,t}]\!] = h^2K^2\varphi_M$, therefore, according to Proposition 3.5(3) this family contains all the good brackets on M for all K. We will consider $f_{h,t}$ as a linear map of $\mathbb{C}^2$ with the coordinates (h, t) to the space $(\Lambda^2\mathfrak{m})^{\mathfrak{l}}$.

Denote by $\mathcal{C}^k(m)$ the space of homogeneous maps $\mathbb{C}^2 \to (\Lambda^k\mathfrak{m})^{\mathfrak{l}}$ of degree m. Thus, $f_{h,t} \in \mathcal{C}^2(1)$. Define the differential $\delta : \mathcal{C}^\bullet(m) \to \mathcal{C}^\bullet(m+1)$ as $(\delta(a))(h, t) = [\![f_{h,t}, a(h, t)]\!]$.

Proposition 4.3.

1. *For all* $(h, t) \in \mathbb{C}^2 \setminus 0$,

$$H^2(M, \delta_{f_{h,t}}) = H^2(M).$$

2. *Let* $m \geq 0$ *and* $b \in \mathcal{C}^3(m+1)$ *such that* $\delta b = 0$. *Then there exists* $a \in \mathcal{C}^2(m)$ *such that* $\delta a = b$.

Proof. Let us prove (1). If $h \neq 0$ then $f_{h,t} \in X_{hK}$ with $hK \neq 0$. Hence, $H^2(M, \delta_{f_{h,t}}) = H^2(M)$ by Proposition 4.1(2). If $h = 0$ then $t \neq 0$ and $f_{0,t}$ is a non-degenerate Poisson bracket on M, therefore, $H^2(M, \delta_{f_{0,t}}) = H^2(M)$ by Proposition 4.1(1).

Let us prove (2). Denote by $\mathcal{L}(m)$ a linear holomorphic vector bundle of degree m over the Riemann sphere $\mathbb{S}$. The space of global sections of $\mathcal{L}(m)$ may be naturally identified with the space of homogeneous polynomials of two variables of degree m. Taking as the variables h and t, one can consider the space $\mathcal{C}^k(m)$ as the space of global sections of a vector bundle, $\mathcal{E}^k(m)$, that is a direct sum of $\dim(\Lambda^k\mathfrak{m})^{\mathfrak{l}}$ copies of $\mathcal{L}(m)$. It is obvious that $f_{h,t}$ defines a map of sheaves, $\delta^k_m : \mathcal{E}^k(m) \to \mathcal{E}^{k+1}(m+1)$. Denote $\mathcal{H}^k(m) = \mathrm{Ker}(\delta^k_m)/\mathrm{Im}(\delta^{k-1}_{m-1})$.

It follows from (1) that $\delta^2_m : \mathcal{E}^2(m) \to \mathcal{E}^3(m+1)$ is a map of bundles, i.e., its image is a direct sub-sheaf of $\mathcal{E}^3(m+1)$. Hence, $\mathcal{H}^3(m+1)$ is a sheaf over $\mathbb{S}$ without torsion. But over a neighborhood of the point of $\mathbb{S}$ with homogeneous coordinates $(0, t)$ $\mathcal{H}^3(m+1) = 0$. This follows from the fact that $f_{0,t}$ is a non-degenerate Poisson bracket and, therefore, $H^3(M, \delta_{f_{0,t}}) = 0$. Since $\mathbb{S}$ is connected, $\mathcal{H}^3(m+1) = 0$ over $\mathbb{S}$. So, $\mathrm{Ker}\,\delta^3_{m+1} = \mathrm{Im}\,\delta^2_m$.

Consider the exact sequence of sheaves over $\mathbb{S}$:

$$0 \to \mathcal{H}^2(m) \to \mathcal{E}^2(m) \to \mathrm{Im}\,\delta^2_m \to 0.$$

To complete the proof of (2), one needs to show that any global section of $\mathrm{Im}\,\delta^2_m$ can be lifted to a global section of $\mathcal{E}^2(m)$. But this follows from the fact that $H^1(\mathbb{S}, \mathcal{H}^2(m)) = 0$. To prove the last fact we observe that $\mathcal{H}^2(m)$ is a sub-bundle of $\mathcal{E}^2(m)$, therefore, is a direct sum of a number of copies of $\mathcal{L}(m)$. Since $m \geq 0$, $H^1(\mathbb{S}, \mathcal{L}(m)) = 0$. This implies that $H^1(\mathbb{S}, \mathcal{H}^2(m) = 0$, too. □

5. The G invariant Φ_h associative quantization in one parameter

Denote by $(HC_M{}^\bullet, \partial)$ the Hochschild complex on M, where each space HC^k_M consists of holomorphic k-differential operators on M. Let X be a complex analytic manifold. The map $\psi : X \to HC^k_M$ is called to be holomorphic if for any open subsets $U \subset X$ and $V \subset M$ and any holomorphic functions $a(x, y)_1, \dots , a(x, y)_k$ on $U \times V\, \psi(a_1, \dots , a_k)$ is also a holomorphic function on $U \times V$. We will denote the map ψ by ψ_x, $x \in X$, and call ψ_x a holomorphic family of k-differential operators on M.

Denote by $HC^k_M(X)$ the space of all holomorphic maps $X \to HC^k_M$ and by $(HC_M{}^\bullet(X), \partial)$ the corresponding complex. It is clear that $(HC_M{}^\bullet(X), \partial)$ is naturally identified with the sub-complex of $(HC_{X\times M}{}^\bullet, \partial)$ consisting of polydifferential operators along M. Denote by $\Lambda^k T_M(X)$ the space of analytic maps of X to the space of polyvector fields on M.

Proposition 5.1. *Let X be a Stein manifold. Then*

$$H^k(HC_M{}^\bullet(X), \partial) = \Lambda^k T_M(X).$$

Proof. The proof can be proceed in the similar way as for $(HC_M{}^\bullet, \partial)$, using that M and X are Stein manifolds [13]. □

Proposition 5.2. *Let $\mathfrak{g}$ be a simple Lie algebra, M a semisimple orbit in $\mathfrak{g}^*$. Let $X = X_{K^2=-1}$ be the manifold of invariant brackets f satisfying $[\![f, f]\!] = -\varphi_M$. Then there exists a holomorphic family of multiplications $\mu_{f,h}$ on $\mathcal{A}$ of the form*

$$\mu_{f,h}(a, b) = ab + \left(\frac{h}{2}\right) f(a, b) + \sum_{n\geq 2} h^n \mu_{f,n}(a, b), \quad f \in X, \tag{5.1}$$

that is, $U(\mathfrak{g})$ invariant and Φ_h associative.

Proof. The proof is essentially follows to [4, Proposition 5.1], but here we construct the multiplication for all f simultaneously using parameterized Poisson cohomologies from the previous section.

To begin, consider the multiplication $\mu^{(1)}(a, b) = ab + (\frac{1}{2}h) f(a, b)$. The corresponding obstruction cocycle is given by

$$\mathrm{obs}_2 = \frac{1}{h^2}(\mu^{(1)}(\mu^{(1)} \otimes \mathrm{id}) - \mu^{(1)}(\mathrm{id} \otimes \mu^{(1)})\Phi_h)$$

considered modulo terms of order h. No $1/h$ terms appear because f is a biderivation and, therefore, a Hochschild cocycle. The fact that the presence of Φ_h does not interfere with the cocycle condition and that this equation defines a Hochschild 3-cocycle was proven in [10] (see the Proof of Proposition 4.1 there). By Proposition 5.1 the differential Hochschild cohomology of $\mathcal{A}$ in dimension p is the space of holomorphic families of p-polyvector fields on M parameterized by X. Since $\mathfrak{g}$ is reductive, the subspace of $\mathfrak{g}$ invariants splits off as a sub-complex and has cohomology given by $(\Lambda^p\mathfrak{m})^{\mathfrak{l}}(X)$. The complete anti-symmetrization of a p-tensor projects the space of invariant differential p-cocycles onto the subspace $(\Lambda^p\mathfrak{m})^{\mathfrak{l}}(X)$ representing the cohomology. The equation $[\![f, f]\!] + \varphi_M = 0$ implies that the obstruction cocycle is a coboundary, and we can find a 2-cochain $\mu_{f,2}$, so that $\mu^{(2)} = \mu^{(1)} + h^2\mu_{f,2}$ satisfies

$$\mu^{(2)}(\mu^{(2)} \otimes \mathrm{id}) - \mu^{(2)}(\mathrm{id} \otimes \mu^{(2)})\Phi_h = 0 \bmod h^3.$$

Assume we have defined the deformation $\mu^{(n)}$ to order h^n such that Φ_h associativity holds modulo h^{n+1}, then we define the $(n+1)$th obstruction cocycle by

$$\mathrm{obs}_{n+1} = \frac{1}{h^{n+1}}(\mu^{(n)}(\mu^{(n)} \otimes \mathrm{id}) - \mu^{(n)}(\mathrm{id} \otimes \mu^{(n)})\Phi_h) \bmod h.$$

In [10, Proposition 4.1], it is shown that the usual proof that the obstruction cochain satisfies the cocycle condition carries through to the Φ_h associative case. The coboundary of obs_{n+1} appears as the h^{n+1} coefficient of the signed sum of the compositions of $\mu^{(n+1)}$ with obs_{n+1}. The fact that $\Phi_h = 1 \bmod h^2$ together with the pentagon identity implies that the sum vanishes identically, and thus all coefficients vanish, including the coboundary in question. Let $\mathrm{obs}'_{n+1} \in (\Lambda^3\mathfrak{m})^{\mathfrak{l}}(X)$ be the projection of obs_{n+1} on the totally skew-symmetric part, which represents the cohomology class of the obstruction cocycle. The coefficient of h^{n+2} in the same signed sum, when projected on the skew symmetric part, is $[\![f, \mathrm{obs}'_{n+1}]\!]$ which is the coboundary of obs'_{n+1}, in the complex $((\Lambda^\bullet\mathfrak{m})^{\mathfrak{l}}(X), \delta = [\![f, \cdot]\!])$. Thus obs'_{n+1} is a δ cocycle. By Proposition 4.2, this complex has zero 3-cohomology. Now we modify μ^{n+1} by adding a term $h^n\mu_{f,n}$ with $\mu_{f,n} \in (\Lambda^2\mathfrak{m})^{\mathfrak{l}}$ and consider the $(n+1)$th obstruction cocycle for $\mu'(n+1) = \mu^{(n+1)} + h^n\mu_{f,n}$. Since the term we added at degree h^n is a Hochschild cocycle, we do not introduce a h^n term in the calculation of $\mu^n(\mu^{(n)} \otimes \mathrm{id}) - \mu^{(n)}(\mathrm{id} \otimes \mu^{(n)})\Phi_h$ and the totally skew-symmetric projection h^{n+1} term has been modified by $[\![f, \mu_{f,n}]\!]$. By choosing $\mu_{f,n}$ appropriately, we can make the $(n+1)$th obstruction cocycle represent the zero cohomology class, and we are able to continue the recursive construction of the desired deformation. □

Let X be as in Proposition 5.2. Let $(o, \mathbb{C}[[h]])$ be the formal manifold which is the formal neighborhood of $0 \in \mathbb{C}$. We call a morphism $\pi : o \to X$ *a formal path* in X. A formal path π may be given by a formal series in h,

$$f(h) = f_0 + hf_1 + h^2 f_2 + \cdots, \quad f_k \in (\Lambda^2\mathfrak{m})^{\mathfrak{l}}$$

satisfying $[\![f(h), f(h)]\!] = -\varphi_M$. It is clear that $f_0 \in X$; we call it the origin of the path $f(h)$. The element f_1 belongs to the tangent space to X at the point f_0, which consists of elements v such that $[\![f_0, v]\!] = 0$.

It is clear that if we put $f = f(h)$ in the multiplication $\mu_{f,h}$, we obtain a $U(\mathfrak{g})$ invariant Φ_h associative multiplication which depends only on h and has the form

$$\mu_{f(h),h}(a,b) = ab + \binom{h}{2} f_0(a,b) + \cdots$$

with φ_M-bracket f_0. In particular, we obtain the following corollary.

Corollary 5.1. *Any φ_M-bracket on a semisimple orbit in $\mathfrak{g}^*$ can be quantized.*

Proof. Indeed, let f_0 be a φ_M-bracket. The multiplication corresponding to any path $f(h) = f_0 + hf_1 + \cdots$ with origin in f_0 is as required. □

Proposition 5.3. *The multiplication $\mu_{f,h}$ of the form* (5.1) *has the following universal properties*:

1. *For any $U(\mathfrak{g})$ invariant Φ_h associative multiplication m_h, there exists a formal path in X, $f(h)$, such that m_h is equivalent to $\mu_{f(h),h}$.*
2. *Multiplications corresponding to different paths are not equivalent.*

We will need the following lemma.

Lemma 5.1. *Let $m_h(a,b) = ab + hf_0(a,b) + h^2 m_2(a,b) + \cdots$ be a multiplication and $f(h)$ a path such that the multiplication $\mu_{f(h),h}$ coincides with m_h modulo h^{n+1}. Then there exist a path $f'(h) = f(h) + h^n p_1 + \cdots$ and a differential operator D such that the multiplication $m'_h = (1 + h^{n+1}D) \circ m_j$,*

$$m'_h(a,b) = (1 + h^{n+1}D)^{-1} m_h((1 + h^{n+1}D)a, (1 + h^{n+1}D)b),$$

coincides with $\mu_{f'(h),h}$ modulo h^{n+2}.

Proof. We have

$$\mu_{f(h),h} = ab + hf_0 + h^2 m_2 + \cdots + h^n m_n + h^{n+1}\mu_{n+1} + \cdots,$$

where m_k, $k = 2, 3, \ldots$, are terms appearing in the expansion of m_h. It is easy to check that $\mu_{n+1} - m_{n+1}$ is a Hochschild cocycle, because both μ_{n+1} and m_{n+1} resolve the same obstruction obs_{n+1},

$$\mathrm{obs}_{n+1}(a,b,c) = \sum_{\substack{i+j=n\\ i,j\geq 1}} (m_i(m_j(a,b),c) - m_i(a, m_j(b,c))),$$

depending only on m_k, $k \leq n$. Hence, one has $\mu_{n+1} = m_{n+1} + \partial D + p_1$, where D is a Hochschild 1-chain, i.e., a differential operator, and p_1, is a bivector field. Applying $1 + h^{n+1}D$ to m_h, we obtain

$$(1 + h^{n+1}D) \circ m_h = \mu_{f(h),h} - h^{n+1} p_1 \bmod h^{n+2}.$$

Observe now that p_1 is a δ_{f_0} cocycle, i.e., $[\![f_0, p_1]\!] = 0$. This follows from the fact that $(1 + h^{n+1}D) \circ m_h$ is a Φ_h associative multiplication. Indeed, if $[\![f_0, p_1]\!]$ is not equal to zero, its contribution to obs_{n+2} is not a Hochschild coboundary. So, p_1 is a tangent vector to X at f_0. Since, by Proposition 3.3(2), X is without singularities, there exists a formal path in X of the form $p(h) = f_0 + h^n p_1 + \cdots$. Let $f'(h)$ be the path that in local coordinates on a neighborhood of f_0 in X is the sum $f(h) + p(h)$. It is clear that $f'(h) = f(h) + h^n p_1 + \cdots$. Putting in $\mu_{f,h} f'(h)$ instead $f(h)$ does not change the coefficients by h^k, $k \le n$, in $\mu_{f(h),h}$ and changes the $(n+1)$th coefficient adding p_1 to it. So, we have

$$m'_h = (1 + h^{n+1} D) \circ m_h = \mu_{f'(h),h} \bmod h^{n+2},$$

as required. □

Proof of Proposition 5.6. Let us prove (1). Let f_0 be the φ_M-bracket corresponding to the multiplication m_h. By Corollary 5.1, one can assume that m_h coincides modulo h^2 with $\mu_{f(h),h}$ for the trivial path $f(h) = f_0$. Using Lemma 5.1, we find D_2 and p_1 such that the multiplication $m_h^{(2)} = (1 + h^2 D_2) \circ m_h$ is equal modulo h^3 to the multiplication corresponding to a path $f_1(h) = f_0 + hp_1 + \cdots$. Now we may apply the lemma to $m_h^{(2)}$, and so on. On the nth step we obtain $m_h^{(n)} = (1 + h^n D_n) \cdots (1 + h^2 D_2) \circ m_h$ that corresponds modulo h^{n+1} to a path $f_{n-1}(h) = f_0 + hp_1 + \cdots + h^{n-1} p_{n-1} + \cdots$. Let $D = \lim(1 + h^n D_n) \cdots (1 + h^2 D_2)$, $f(h) = \lim f_{n-1}(h)$ in h-adic topology. It is clear that such limits exist. We obtain that $D \circ m_h = \mu_{f(h),h}$, which proves (1).

The proof of (2) using $(\Lambda^1 \mathfrak{m})^{\mathfrak{l}} = 0$ is left to the reader. □

Remark 5.1. Let X_0 be the variety of all non-degenerate Poisson brackets on M (see (3.21)). As in the proof of Proposition 5.2, one can construct a holomorphic family of $U(\mathfrak{g})$ invariant associative multiplications of the form

$$\mu_{p,t}(a,b) = ab + \binom{t}{2} p(a,b) + \sum_{n \ge 2} t^n \mu_{p,n}(a,b), \quad p \in X_0. \tag{5.2}$$

The same argument as in the proof of Proposition 5.3 shows that such a family has the universal property: any $U(\mathfrak{g})$ invariant deformation quantization on M is equivalent to the pullback of (5.2) by a unique formal path in X_0.

6. The G invariant Φ_h associative quantization in two parameters

Proposition 6.1. *Let $\mathfrak{g}$ be a simple Lie algebra, M a semisimple orbit in $\mathfrak{g}$. Let v be the KKS Poisson bracket on M. Let f be an invariant bracket on M satisfying* $[\![f, f]\!] = -\varphi_M$, $[\![f, v]\!] = 0$. *Then there exists a two parameter multiplication $\mu_{t,h}$ on $\mathcal{A}$:*

$$\mu_{t,h}(a,b) = ab + \binom{h}{2} f(a,b) + \binom{t}{2} v(a,b) + \sum_{k+l \ge 2} h^k t^l \mu_{k,l}(a,b), \tag{6.1}$$

which is $U(\mathfrak{g})$ invariant and Φ_h associative.

Proof. The existence of a multiplication which is Φ_h associative up to and including h^2 terms is nearly identical to the proof of Proposition 5.2.

So, suppose we have a multiplication defined to order n,

$$\mu_{t,h}^{(n)}(a,b) = ab + \left(\frac{h}{2}\right) f(a,b) + \left(\frac{t}{2}\right) v(a,b) + \sum_{2\le k+l\le n} h^k t^l \mu_{k,l}(a,b),$$

which is $U(\mathfrak{g})$ invariant and Φ_h associative to order h^n. Consider the obstruction cochain,

$$\mathrm{obs}_{n+1} = \sum_{k=0,\dots,n+1} h^k t^{n+1-k} b_k.$$

The same argument as in the proof of Proposition 5.2 shows that obs_{n+1} is a Hochschild cocycle. This means that all coefficients b_k are Hochschild cocycles. Hence, $b_k = \partial a_k + \beta_k$ for all k, where $\beta_k \in (\Lambda^3\mathfrak{m})^{\mathfrak{l}}$. Therefore,

$$\mathrm{obs}_{n+1} = \partial a + \beta,$$

where $a = \sum h^k t^{n+1-k} a_k,\ \beta = \sum h^k t^{n+1-k}\beta_k$. The element β is a cocycle from $\mathcal{C}^3(n+1)$ (see Section 4.4). By Proposition 4.3(2) there exists $\alpha \in \mathcal{C}^2(n)$ such that $[\![hf + tv, \alpha]\!] = \beta$. This shows that, as in the proof of Proposition 5.2, we can modify $\mu_{t,h}^{(n)}$ adding a multiple of α to get a new multiplication to order n with $(n+1)$th obstruction cocycle obs_{n+1}, being a Hochschild coboundary ∂a. So, we are able to continue the recursive construction of the desired two parameter deformation. □

Remark 6.1. Let $\pi : Y \to X_0$ be the projection of the variety of good brackets over M to the variety of non-degenerate Poisson brackets (see Remark 3.2). In the similar way as Proposition 6.1, one can prove the existence of a family of multiplications of the form

$$\mu_{f,t,h}(a,b) = ab + \left(\frac{h}{2}\right) f(a,b) + \left(\frac{t}{2}\right)(\pi f)(a,b) + \sum_{k+l\ge 2} h^k t^l \mu_{f,k,l}(a,b),\quad f \in Y. \tag{6.2}$$

This family satisfies the universal property for two parameter quantizations, i.e., any two parameter $U(\mathfrak{g})$ invariant Φ_h associative quantization on M of the form (6.1) is the pullback of a two parameter formal path in Y.

7. Polarization

We retain notations from the previous sections. Recall that $M = G/L$ where G is a complex connected simple Lie group and L is a Levi subgroup. It is known that the natural projection $\pi : G \to M$ is a holomorphic principal fiber bundle with structure group L.

The tangent space to M, $T(M)$, is the associated vector bundle corresponding to the ad-action of L on $\mathfrak{m}$, a unique $\mathfrak{l}$ invariant subspace in $\mathfrak{g}$ complement to $\mathfrak{l}$ (see Section 3.5). According to Proposition 3.1, $T(M)$ may be presented as a direct sum of sub-bundles, $T(M) = \oplus_{\bar\alpha\in\bar\Omega} T_{\bar\alpha}(M)$ where $T_{\bar\alpha}(M)$ is the associated vector bundle corresponding to $\mathfrak{m}_{\bar\alpha}$.

Assigning to each $g \in G$ the horizontal subspace $g\mathfrak{m}$ provides G with an invariant connection ∇. This connection defines a G invariant connection on any associated vector bundle over M.

Let us choose $\bar{\Omega}^+$, a system of positive quasiroots in $\bar{\Omega}$. Let the sub-bundle $T^+(M) = T_{\bar{\Omega}^+}(M)(T^-(M) = T_{\bar{\Omega}^-}(M))$ correspond to $\mathfrak{m}^+ = \oplus_{\bar{\alpha}\in\bar{\Omega}^+} \mathfrak{m}_{\bar{\alpha}} (\mathfrak{m}^- = \oplus_{\bar{\alpha}\in\bar{\Omega}^-} \mathfrak{m}_{\bar{\alpha}})$.

By a realization M as an orbit in $\mathfrak{g}^*$ the decomposition $T(M) = T_{\bar{\Omega}^+}(M) \oplus T_{\bar{\Omega}^-}(M)$ defines complement polarizations of M with respect to the KKS symplectic form on M. These polarizations define two complement foliations on M, which are fibrating with the natural projections $M \to G/P^+$ and $M \to G/P^-$, respectively, where P^+, P^- are upper and lower parabolic subgroups containing L.

Let $\mathcal{A}^+ = \mathcal{A}_{\bar{\Omega}^+}$ denote the sheaf of holomorphic functions a on M constant along the polarization defined by $\bar{\Omega}^+$, i.e., such that $\nabla_X a = 0$ for any vector field $X \in T_{\bar{\Omega}^+}(M)$.

Proposition 7.1. *Let ν be a $U(\mathfrak{g})$ invariant bidifferential operator on M vanishing on constants. Let $\bar{\Omega}^+$ be a system of positive quasiroots. Then $\nu(a, b) = 0$ for any sections $a, b \in \mathcal{A}_{\bar{\Omega}^+}$.*

Proof. The connection ∇ induces an equivariant isomorphism of $\mathcal{A}$ modules between the sheaf of differential operators on $\mathcal{A}$ and the sheaf $ST^-(M) \otimes_{\mathcal{A}} ST^+(M)$, where $ST^-(M)$ and $ST^+(M)$ denote the sheaves of symmetric tensors over $T^-(M)$ and $T^+(M)$. It provides an equivariant isomorphism between the space of invariant bidifferential operators on M and the space $((S\mathfrak{m}^- \otimes_{\mathbb{C}} S\mathfrak{m}^+)^{\otimes 2})^{\mathfrak{l}}$. Thus one may regard ν as a sum of terms of the form $A_1B_1 \otimes A_2B_2$ where $A_1, A_2 \in S\mathfrak{m}^-,\ B_1, B_2 \in S\mathfrak{m}^+$. Since ν is invariant, $A_1B_1 \otimes A_2B_2$ must be of weight zero with respect to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{l}$. Since $A_1B_1 \otimes A_2B_2$ is vanishing on constants, either B_1 or B_2 must belong to a positive symmetric power of $\mathfrak{m}^+$, let such be B_1. But the corresponding to B_1 differential operator takes the functions of $\mathcal{A}^+$ to zero, therefore, the bidifferential operator corresponding to $A_1B_1 \otimes A_2B_2$ when applies to the pair $a, b \in \mathcal{A}^+$ gives zero, too. □

Corollary 7.1. *All $U(\mathfrak{g})$ invariant Φ_h associative multiplications are trivial on $A_{\bar{\Omega}^+}$ for any choice of $\bar{\Omega}^+$. It means that for any of such a multiplication, μ, one has $\mu(a, b) = ab$ whenever $a, b \in \mathcal{A}_{\bar{\Omega}^+}$.*

Proof. Indeed, μ has the form $\mu(a, b) = ab + \{$bidifferential operators vanishing on constants$\}$. So, the corollary follows from Proposition 7.1. □

Since μ from Corollary 7.1 when restricted to $\mathcal{A}^+$ coincides with the usual multiplication, it is associative in the usual sense. On the other hand, μ is Φ_h associative, so the usual and Φ_h associativities coincide on $\mathcal{A}^+$. We will prove this fact independently in a more general setting.

Let V be a representation of L. Denote by $V(M)$ the corresponding associated vector bundle on M. When this does not lead to confusion we will use the same notation $V(M)$ for the sheaf of holomorphic sections of the bundle $V(M)$. Denote by $V^+(M) = V_{\bar{\Omega}^+}(M)$

the sheaf of holomorphic sections v of $V(M)$ constant along the polarization defined by $\bar{\Omega}^+$, i.e., such that $\nabla_X v = 0$ for any vector field $X \in T_{\bar{\Omega}^+}(M)$.

It is clear that $V(M)$ is an $\mathcal{A}$ module under the natural multiplication $m : \mathcal{A} \otimes V(M) \to V(M),\ m(a, v) = av,\ a \in \mathcal{A},\ v \in V(M)$. Since $\mathcal{A}$ is commutative, $V(M)$ may be considered as a two-sided module. This multiplication is obviously associative, i.e., for sections $a, b \in \mathcal{A}$ and $v \in V(M)$ one has $(ab)v = a(bv)$ and $(av)b = a(vb)$. The following proposition shows that this multiplication being restricted to $\mathcal{A}^+$ and $V^+(M)$ is also Φ_h associative.

Proposition 7.2. *Let V be a representation of L. Let us choose a system of positive quasiroots $\bar{\Omega}^+$. Then for any sections $a, b \in \mathcal{A}^+,\ v \in V^+(M)$ one has $abv = m\Phi_h(a \otimes b \otimes v)$ and $avb = m\Phi_h(a \otimes v \otimes b)$.*

Proof. Let us prove the first relation. The second relation can be proven similarly. Let Ω^+ be a system of positive roots for $\mathfrak{g}$ which projects on $\bar{\Omega}^+$. Let $\mathfrak{p}^+$ be the corresponding parabolic subalgebra of $\mathfrak{l}$ and $\mathfrak{u}^+$ the radical of $\mathfrak{p}^+$, so $\mathfrak{p}^+ = \mathfrak{l} \oplus \mathfrak{u}^+$. Let $\mathfrak{p}^-$ and $\mathfrak{u}^-$ denote the corresponding opposite subalgebras, in particular, $\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{u}^-$ and $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{u}^+$.

Let $\pi : G \to M$ be the natural projection. Let U be an open set in M. Sections of $V(M)$ over U can be identified with functions $f : \pi^{-1}(U) \to V$ such that $f(gl) = l^{-1}f(g)$ for $l \in L,\ g \in G$. Sections of $V^+(M)$ must, in addition, satisfy the condition $E_\alpha f = 0,\ a \in \mathfrak{u}^+$, and the root vector E_α acts on f as a complex left-invariant vector field on G. In particular, functions of $\mathcal{A}^+$ are identified with functions ψ over π^{-1} such that $E_\alpha\psi = 0$ for all $\alpha \in \mathfrak{p}^+$.

Let us write Φ_h in the form

$$\Phi_h = 1 \otimes 1 \otimes 1 + \sum_{k \geq 2} h^k \Phi_k^1 \otimes \Phi_k^2 \otimes \Phi_k^3, \tag{7.1}$$

where each $\Phi_k^i,\ i = 1, 2$, belongs to $S\mathfrak{p}^+ \otimes S\mathfrak{u}^-$ and Φ_k^3 belongs to $S\mathfrak{p}^- \otimes S\mathfrak{u}^+$ ($S\mathfrak{p}^+$ denote the space of symmetric tensors over $\mathfrak{p}^+$, and so on). The total degree of each $\Phi_k^1 \otimes \Phi_k^2 \otimes \Phi_k^3$ is greater than zero.

Let us take $a, b \in \mathcal{A}^+,\ v \in V^+(M)$ and apply $\Phi_k^1 \otimes \Phi_k^2 \otimes \Phi_k^3$ to $a \otimes b \otimes v$. Suppose $\Phi_k^1(a)$ and $\Phi_k^2(b)$ are not equal to zero. Then there are $x_1, x_2 \in S\mathfrak{u}^-$ such that $\Phi_k^1 = A \otimes x_1,\ \Phi_k^2 = B \otimes x_2$, where $A, B \in S\mathfrak{p}^+$. But since $\Phi_k^1 \otimes \Phi_k^2 \otimes \Phi_k^3$ is an invariant element, it must be of degree zero under the Cartan subalgebra. It follows that Φ_k^3 has to be of the form $\Phi_k^3 = C \otimes x_3$ where $C \in S\mathfrak{p}^-,\ x_3 \in \mathfrak{u}^+$ and $x_3 \neq 0$. Hence, $\Phi_k^3(v) = 0$.

So, we have proven that in the expression (7.1) all terms except for the first when applying to $a \otimes b \otimes v$ are equal to zero. □

8. The real case

Let G be a real connected simple Lie group with complexification $G^{\mathbb{C}}$. Let $g_{\mathbb{R}}$ be the Lie algebra of G with complexification $\mathfrak{g}$. Let L be the stabilizer of a semisimple element $\lambda \in \mathfrak{g}_{\mathbb{R}}^*$, so that $M = G/L$ may be identified with the coadjoint orbit passing through λ. It is well known that L is connected, therefore the complexification $L^{\mathbb{C}} \subset G^{\mathbb{C}}$ is meaningful.

Denote by $\mathfrak{l}_{\mathbb{R}}^{*}$, ($\mathfrak{l}$) the Lie algebra of L, ($L^{\mathbb{C}}$). Note that $\mathfrak{l}$ is a Levi subalgebra in $\mathfrak{g}$. The natural embedding $M \to M^{\mathbb{C}} = G^{\mathbb{C}}/L^{\mathbb{C}}$ may be regarded as complexification of M.

Let $C^{a}(M)$ and $C^{\infty}(M)$ denote the sheaves of real analytic and smooth complex-valued functions on M. The action of G on M defines a map of $\mathfrak{g}_{\mathbb{R}}$ into the Lie algebra of real vector fields on M, $\mathfrak{g}_{\mathbb{R}} \to \mathrm{Vect}_{\mathbb{R}}(M)$, that extends to a map $\mathfrak{g} \to \mathrm{Vect}(M)$ of $\mathfrak{g}$ into the Lie algebra of complex vector fields on M. It follows that $U(\mathfrak{g})$ acts on the sections of $C^{a}(M)$ and $C^{\infty}(M)$ as differential operators.

As a consequence we get that all the $U(\mathfrak{g})$ invariant Φ_h associative multiplications constructed in the previous sections can be defined on the real manifold M.

Let us choose a system of positive roots, Ω^{+}, in $\mathfrak{g}$. Let P be the corresponding parabolic subgroup of $G^{\mathbb{C}}$ with Levi factor $L^{\mathbb{C}}$ and $\mathfrak{p}$ its Lie algebra. One has $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}^{+}$, where $\mathfrak{u}^{+}$ is the nil radical of $\mathfrak{p}$. We assume that $\mathfrak{p}$ is θ-stable parabolic, i.e., satisfies the condition

$$\mathfrak{g}_{\mathbb{R}} \cup \mathfrak{p} = \mathfrak{l}_{\mathbb{R}}. \tag{8.1}$$

Then the natural map

$$M = G/L \to G^{\mathbb{C}}/P \tag{8.2}$$

is an inclusion and the image is an open set. Thus the choice of Ω^{+} makes M into a complex manifold with holomorphic action of G. The corresponding system of positive quasiroots, $\bar{\Omega}^{+}$, defines a complex polarization on M, whereas $\bar{\Omega}^{-}$ defines the complement polarization.

Note that for $\mathfrak{l}_{\mathbb{R}}$ a θ-stable parabolic $\mathfrak{p}$ exists if $\mathfrak{l}_{\mathbb{R}}$ is the centralizer of a semisimple element $x \in \mathfrak{g}_{\mathbb{R}}$ such that $\mathrm{ad}(x)$ has imaginary eigenvalues.

One can prove [15] that the smooth functions on M which are constant along the polarizations defined by $\bar{\Omega}^{+}$ and $\bar{\Omega}^{-}$ coincide with holomorphic and anti-holomorphic functions on M.

Let $\widetilde{S}$ denote the operator $F_h \sigma F_h^{-1}$, where F_h is from Proposition 2.1 and σ is the usual permutation, acting on the tensor product of any two representations of $U(\mathfrak{g})$. Let B_h be a quantized algebra of functions on M. We say that the multiplication on B_h, m_h, is $\widetilde{S}$-commutative, if for any $a, b \in B$ one has $m_h(a \otimes b) = m_h \widetilde{S}(a \otimes b)$.

Theorem 8.1. *Let G be a real connected simple Lie group, L a Lie subgroup which is a stabilizer of a semisimple element* $\lambda \in \mathfrak{g}_{\mathbb{R}}^{*}$, *and* $M = G/L$. *Let* $r \in \wedge^{2}\mathfrak{g}$ *be an r-matrix and* $U_h(\mathfrak{g})$ *the corresponding quantum group. Let X be the variety of φ-brackets on M, as in* Proposition 5.2, *for* $\varphi = [\![r, r]\!]$. *Then there exists a universal family of multiplications on* $C^{\infty}(M)$ *of the form*

$$m_{f,h}(a, b) = ab + \left(\frac{h}{2}\right)(f - r_M)(a, b) + \sum_{n \geq 2} m_{f,n}(a, b), \quad f \in X, \tag{8.3}$$

which is $U_h(\mathfrak{g})$ *invariant and associative.*

Suppose the map (8.2) *induces a complex structure on M. Then* $m_{f,h}$ *being restricted to the sheaf of holomorphic functions is* $\widetilde{S}$ *commutative.*

Proof. According to Proposition 2.2, we put $m_{f,h} = \mu_{f,h} F_h^{-1}$, where the multiplication $\mu_{f,h}$ is from Proposition 5.2. The functions of $C^a(M)$ are restrictions to M of holomorphic functions on $M^{\mathbb{C}}$. It follows that $m_{f,h}$ is a well-defined multiplication on $C^a(M)$. Since the coefficients by h^n in $m_{f,h}$ are bidifferential operators, this multiplication is defined, actually, for smooth complex-valued functions on M. The universality of $m_{f,h}$ follows from the universality of $\mu_{f,h}$ (see Proposition 5.3). The $\widetilde{S}$-commutativity of $m_{f,h}$ for holomorphic functions follows directly from the commutativity of $\mu_{f,h}$ (for such functions, see Corollary 7.1). □

As a consequence, we obtain the statement reverse to Corollary 2.1: any Poisson bracket on M of the form (2.8) admits a $U_h(\mathfrak{g})$ invariant quantization. The proof is analogous to Corollary 5.1.

The following theorem contains, in particular, the statement reverse to Corollary 2.2.

Theorem 8.2. *Let M be as in Theorem* 8.1. *Let v be the KKS Poisson bracket on M. Let $r \in \wedge^2\mathfrak{g}$ be an r-matrix and $U_h(\mathfrak{g})$ the corresponding quantum group. Let p be a Poisson bracket on M of the form $p = f - r_M$, where f satisfies $[\![f, f]\!] = [\![r_M|, r_M]\!]$ and $[\![f, r_M]\!] = 0$. Then there exists a two parameter $U_h(\mathfrak{g})$ invariant associative multiplication on $C^\infty(M)$ of the form*

$$m_{t,h}(a, b) = ab + \left(\frac{h}{2}\right)(f - r_M)(a, b) + \left(\frac{t}{2}\right) v(a, b) + \sum_{k+l\geq 2} h^k t^l m_{k,l}(a, b).$$

Suppose the map (8.2) *induces a complex structure on M. Then $m_{f,h}$ being restricted to the sheaf of holomorphic functions is $\widetilde{S}$ commutative*.

Proof. We put $m_{t,h} = \mu_{t,h} F_h^{-1}$ and use the argument as in the proof of Theorem 8.1. □

Remark 8.1. As follows from Corollary 7.1, any $U_h(\mathfrak{g})$ invariant multiplications, in particular the multiplications from Theorems 8.1 and 8.2, being restricted to holomorphic functions are equal to $m_0 F_h^{-1}$, where m_0 is the usual multiplication.

9. The quantization of vector bundles

Let $\rho : L \to GL(V)$ be a representation of L in a complex vector space V. Then ρ extends holomorphically to $L^{\mathbb{C}}$. Let $\rho_{\mathbb{C}}$ denote such an extension. The vector bundle on M, $V(M)$, associated with ρ is the restriction of the vector bundle on $M^{\mathbb{C}}$, $V(M^{\mathbb{C}})$, associated with $\rho_{\mathbb{C}}$. Let us choose a system of positive quasiroots, $\bar{\Omega}^+$, and the corresponding parabolic subgroup, $P \subset L^{\mathbb{C}}$. Let us assume that the map (8.2) defines the complex structure on M. Let ρ_P denote the extension of $\rho_{\mathbb{C}}$ to P, which is trivial on the unipotent radical of P. Then $V(M)$ is the pullback of the vector bundle on $G^{\mathbb{C}}/P$ associated by ρ_P. So, $V(M)$ acquires the structure of a holomorphic vector bundles on M. The holomorphic sections of $V(M)$ form a sheaf $V^+(M)$ whose sections are the restrictions to M of sections of $V^+(M^{\mathbb{C}})$ constant along the polarization defined by $\bar{\Omega}^+$.

In the following, we fix a system $\bar{\Omega}^+$ and the corresponding complex structure on M. Given a representation V of G, we denote by $V(M)$ and $V^+(M)$ the sheaves of smooth and holomorphic sections of the associated to V vector bundle on M.

Definition 9.1. Let $\mathcal{A} \subset C^\infty(M)$ be a sub-sheaf of algebras and E a sheaf of $\mathcal{A}$ modules with a $U(\mathfrak{g})$ action. Let $m_0 : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ and $n_0 : \mathcal{A}\otimes E \to E$ denote the multiplication in $\mathcal{A}$ and the action of $\mathcal{A}$ on E. Let $\mathcal{A}_h$ be a $U_h(\mathfrak{g})$ invariant quantization of $\mathcal{A}$ with the deformed multiplication $m_h = m_0 + hm_1 + o(h)$. We say that $E[[h]]$ is a quantization of E as a $\mathcal{A}_h$ module if a deformed $U_h(\mathfrak{g})$ invariant action $n_h = n_0 + hn_1 + o(h) : \mathcal{A} \otimes E \to E[[h]]$ is given, which makes $E[[h]]$ into an $\mathcal{A}_h$ module. In particular, it means that the associativity holds: $n_h(m_h(a, b), x) = n_h(a, n_h(b, x))$ for $a, b \in \mathcal{A},\ x \in E$.

Since $\mathcal{A}$ is commutative, E is, in fact, a two-sided module. So, in the similar way one defines a quantization of E as a two-sided $\mathcal{A}_h$ module.

Example 9.1. Let $\mathcal{A}^+$ be the sheaf of holomorphic functions on M. Then, as follows from Remark 8.1, there exists a unique $U_h(\mathfrak{g})$ invariant quantization of $\mathcal{A}^+$, $\mathcal{A}_h^+$, and it has the multiplication of the form $m_h = m_0F_h^{-1}$. Let V be a representation of L. Then Proposition 7.2 and the argument of Theorem 8.1 show that the sheaf of holomorphic sections $V^+(M)$ can be uniquely quantized as a left and even as a two-sided module. We denote this quantization by $V_h^+(M)$. The left (right) multiplication by elements of $\mathcal{A}$ has the form $n_h(a \otimes x) = n_0F_h^{-1}(a \otimes x)(n_h(x \otimes a) = n_0F_h^{-1}(x \otimes a))$ for $a \in \mathcal{A},\ x \in E$.

The following proposition shows that the sheaf of smooth sections $V(M)$ admits a $U_h(\mathfrak{g})$ invariant quantization.

Theorem 9.1. *Let $A = C^\infty(M)$ and $\mathcal{A}_h$ be a $U_h(\mathfrak{g})$ invariant quantization of $\mathcal{A}$. Let V be a representation of L. Then there exists a $U_h(\mathfrak{g})$ invariant quantization of V(M) as a left $\mathcal{A}_h$ module*.

Proof. We have $V(M) = \mathcal{A} \otimes_{\mathcal{A}^+} V^+(M)$. Let $V_h(M) = \mathcal{A}_h \otimes_{\mathcal{A}_h^+} V_h^+(M)$, where $\mathcal{A}_h$ is considered as a right and $V_h^+(M)$ as a left $\mathcal{A}_h^+$ module (see Example 9.1). It is clear that $V_h(M)$ is the required quantization. □

9.1. The two-sided quantization

In general, it is not clear whether a two-sided quantization of $V(M)$ exists. However, we will show that there is a quantization, $\mathcal{A}_h$, of the sheaf of smooth functions on M such that for any representation V there exist a quantization of $V(M)$ as a two-sided $\mathcal{A}_h$ module.

Let us construct the quantization $\mathcal{A}_h$. Let

$$R = F_h\, e^{h\mathbf{t}/2} F_h^{-1} = R_i' \otimes R_i'' \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}),$$

where $\mathbf{t} \in \mathfrak{g} \otimes \mathfrak{g}$ is the split Casimir, be the R-matrix (summation by i is assumed). It satisfies the property [8]

$$\Delta'(x) = R\Delta(x)R^{-1}, \quad x \in U_h(\mathfrak{g}),$$

where Δ is the comultiplication in $U_h(\mathfrak{g})$ and Δ' the opposite one,

$$(\Delta \otimes 1)R = R^{13}R^{23} = R'_i \otimes R'_j \otimes R''_i R''_j,$$
$$(1 \otimes \Delta)R = R^{13}R^{12} = R'_i R'_j \otimes R''_j \otimes R''_i,$$

and

$$(1 \otimes \varepsilon)R = (\varepsilon \otimes 1)R = 1 \otimes 1,$$

where ε is the co-unit in $U_h(\mathfrak{g})$.

The element R defines the $U_h(\mathfrak{g})$ equivariant map $S : E \otimes F \to F \otimes E$, $a \otimes b \mapsto \sigma R(a \otimes b)$, σ is the usual permutation, for any $U_h(\mathfrak{g})$ modules E and F.

Let $\mathcal{A}'_h = \mathcal{A}^+_h \otimes_{\mathbb{C}} \mathcal{A}^-_h$, where $\mathcal{A}^-_h$ is a unique $U_h(\mathfrak{g})$ invariant quantization of the sheaf $\mathcal{A}^-$ of anti-holomorphic functions on M. We provide $\mathcal{A}'_h$ with the structure of a sheaf of algebras in the following way. For $a = a_1 \otimes b_1$, $b = a_2 \otimes b_2 \in \mathcal{A}'_h$, we put $m_h(a, b) = a_1a'_2 \otimes b'_1b_2$, where $a'_2 \otimes b'_1 = S(b_1 \otimes a_2)$ and $a_1a'_2$ and b'_1b_2 means the multiplications in A^+_h and A^-_h, respectively. It easily follows from the above properties of R that the multiplication m_h is $U_h(\mathfrak{g})$ invariant and associative on $\mathcal{A}'_h$ and is presented as a power series in h with coefficients being bidifferential operators on $\mathcal{A}$. Since bidifferential operators on smooth functions are fully defined by their values on $\mathcal{A}^+ \otimes \mathcal{A}^-$, this multiplication can be extended to the whole algebra $\mathcal{A}$ of smooth functions on M.

One can show that the Poisson bracket of the obtained quantization $\mathcal{A}_h$ is the bracket reduced to M from the Poisson bracket $r' - r''$ on the group G (see Remark 2.1).

Theorem 9.2. *Let $\mathcal{A}_h$ be the quantization of the sheaf of smooth functions on M constructed above. Let V be a representation of L. Then there exists a quantization of the sheaf of smooth sections of $V(M)$ as a two-sided $\mathcal{A}_h$ module.*

Proof. Let $V'_h(M) = V^+_h(M) \otimes_{\mathbb{C}} \mathcal{A}^-_h$. Let us define left and right multiplications of elements of $V'_h(M)$ by elements of $\mathcal{A}'_h = \mathcal{A}^+_h \otimes_{\mathbb{C}} \mathcal{A}^-_h$. Let $a = a_1 \otimes b_1 \in \mathcal{A}'_h$ and $x = x_1 \otimes b_2 \in V'_h(M)$. Put $n^{\mathrm{left}}_h(a \otimes x) = a_1x'_1 \otimes b'_1b_2$, where $x'_1 \otimes b'_1 = S(b_1 \otimes x_1)$, and $n^{\mathrm{right}}_h(x \otimes a) = x_1a'_1 \otimes b'_2b_1$, where $a'_1 \otimes b'_2 = S(b_2 \otimes a_1)$. Here $a_1x'_1$ means, e.g., the multiplication when $V^+_h(M)$ is considered as a left $\mathcal{A}^+_h$ module. It easily follows from the above properties of R that the multiplications n^{left}_h and n^{right}_h make $V'_h(M)$ into a two-sided A'_h module. The same argument as in the proof of Theorem 9.1 shows that those multiplications define, in fact, the structure of a $\mathcal{A}_h$ module on $V(M)[[h]]$. □

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References

[1] A.A. Belavin, V.G. Drinfeld, Triangle Equations and Simple Lie Algebras, Soviet Scientific Review, Vol. 4, Section C, Harwood, New York, 1984, pp. 93–165.
[2] A. Borel, Sur la cohomologie des espaces fibres principaux et des espaces homogénes de groupes de Lie compacts, Ann. Math. 57 (1953) 115–207.
[3] J. Donin, D. Gurevich, QuasiHopf algebras and R-matrix structures in line bundles over flag manifolds, Selecta Math. 12 (1) (1992) 37–48.
[4] J. Donin, D. Gurevich, S. Shnider, Double quantization on some orbits in the coadjoint representations of simple lie groups, Commun. Math. Phys. 204 (1) (1999) 39–60.
[5] J. Donin, Double quantization on coadjoint representations of simple Lie groups and its orbits, math/9909160.
[6] J. Donin, Double quantization on the coadjoint representation of $\mathrm{sl}(n)^*$, Czechoslovak J. Phys. 47 (11) (1997) 1115–1122.
[7] V.G. Drinfeld, On almost-cocommutative Hopf algebras, Leningrad Math. J. 1 (1990) 321–342.
[8] V.G. Drinfeld, QuasiHopf algebras, Leningrad Math. J. 1 (1990) 1419–1457.
[9] V.G. Drinfeld, On quasitriangular quasiHopf algebras and a group closely connected with $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, Leningrad Math. J. 2 (4) (1991) 829–860.
[10] J. Donin, S. Shnider, Quantum symmetric spaces, J. Pure Appl. Algebra 100 (1995) 103–115.
[11] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras I, Selecta Math. 2 (1) (1996) 1–41.
[12] P. Etingof, A. Varchenko, Geometry and classification of solutions of the classical dynamical Yang–Baxter equation, Commun. Math. Phys. 192 (1998) 77–120.
[13] G. Hochschild, B. Kostant, A. Rosenberg, Differential forms on regular affine algebras, Trans. Am. Math. Soc. 102 (1962) 383–406.
[14] A. Joseph, Faithfully flat embeddings for minimal primitive quotients of quantized enveloping algebras. Quantum deformations of algebras and their representations, Israel Math. Conf. Proc. 7 (1993) 79–106.
[15] A.W. Knapp, Introduction to representations in analytic cohomology, Contemp. Math. 154 (1992) 1–19.
[16] S. Khoroshkin, A. Radul, V. Rubtsov, A family of Poisson structures on compact Hermitian symmetric spaces, Commun. Math. Phys. 152 (1993) 299–316.
[17] J.-H. Lu, Classical dynamical R-matrices and homogeneous Poisson structures on G/H and K/T, math/9909004.
[18] S. Shnider, S. Sternberg, Quantum Groups, Int. Press, 1993.